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On the Minimum Sampling Rate of Signals of Non-bandlimited Response

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Abstract

Recently, consistently resampling has been proposed to resampling discrete signals without bandlimit constraint. In this paper, we study the constraints of the resampling rate such that the input discrete signal can be consistently resampled. We approach it through identifying the rate of innovation (RI) of the signal as introduced by innovation sampling. First, an upper bound on the RI for signals in shift invariant spaces is established. The RI of the signal indicates the globally minimum sampling rate for the signal regardless of the choice of sampling function. Then the properties of the sampling filters such that the signal can be sampled at this rate are specified. Further, we extend the choice of sampling functions to a wider set of general functions from the Hilbert space and the locally minimum sampling rate for the function used accordingly. The results obtained herein are used to obtain the minimum resampling rate for consistent resampling.

1. INTRODUCTION

The sampling rate of the discrete signals used in digital system often needs to be increased or decreased according to the requirements of a particular processing stage. To change the sampling rate of a digital signal, a two-step process is involved [1]. First, the original digital signal is converted, conceptually, to analog form. Then this analog signal is resampled at a different sampling rate or at different sampling locations. Sometimes the resampling functions can also be different from the sampling function used to obtain the input sequence. We address such system as *resampling system*. Assuming that both the input and output are of uniform sampling rate, a typical resampling system is shown in Fig. 1.

Referring to Fig. 1, the input $f_T[n]$ is resampled to $f_{T'}[m]$ using the interpolation and resampling function ϕ and ψ. The subscripts T and T' are used to indicate the sampling interval of the sequence. Notice that ϕ and ψ are dilated by their sampling intervals respectively. When $f_T[n]$ is bandlimited, ϕ and ψ are the classic sinc-Diracs pair. The allowable resampling rate T' are governed by Shannon's sampling theorem.

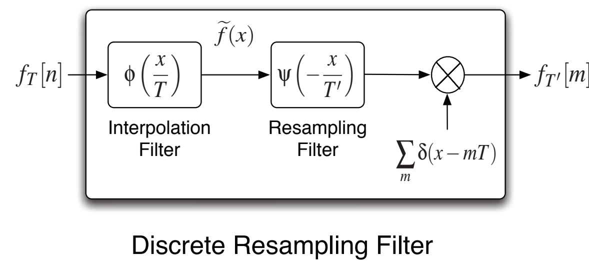

Figure 1: A resampling system with generalized interpolating and resampling functions.

In practice, most signals are not strictly bandlimited. To resample $f_T[n]$ of non bandlimited response, the consistent resampling theory has been developed [2, 3]. The functions ϕ and ψ are normally chosen from the square integrable Hilbert Space $\mathcal{H}$ that satisfying the Riesz condition [9]. For every finite scalar sequence $\{c[k]\}_{k\in\mathbb{Z}}$, the Riesz condition states that [4]

$$A\|c[k]\|^2_{\ell^2} \leq \|\sum_k c[k]\phi(x-k)\|^2_{L^2} \leq B\|c[k]\|^2_{\ell^2} \qquad (1)$$

where A and B are two constants and $0 < A \leq B$. If $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ satisfies (1), the set $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ is admissible and is the *Riesz Basis* of the space V^ϕ. ϕ is referred as the *generating* function of V^ϕ.

Referring to Fig. 1, $f_T[n]$ is said to be consistently resampled if the same continuous function $\widetilde{f}(x)$ can be reconstructed from $f_T[n]$ and $f_{T'}[m]$. Mathematically, let

$$\widetilde{f}_1(x) = \sum_n f_T[n]\phi\left(\frac{x}{T}-n\right) \qquad (2)$$

$$\widetilde{f}_2(x) = \sum_m f_{T'}[m]\phi\left(\frac{x}{T'}-m\right) \qquad (3)$$

Consistent resampling requires $\widetilde{f}_1 = \widetilde{f}_2$. It is optimal in the sense that the input signals can always be reconstructed from the output, i.e., the resampling process is informationally lossless.

The resampling rate for consistent resampling is intuitively subject to constraints related to interpolation as well as the resampling function. Recently, the *rate of innovation*(RI) is introduced to indicate the the minimum sampling rate for a continuous signal [7]. For a signal $f(x)$, its RI is defined by the degree of freedom per unit time and denoted by ρ. The innovation sampling theory states that $f(x)$ can be perfectly reconstructed from its samples of rate ρ by proper acquisition / synthesis functions. For example, the RI of the periodic impulse train

$$f(x) = \sum_{k=0}^{K-1} c_k \sum_{n \in Z} \delta(x - x_k - n\tau) \tag{4}$$

is given by $\rho_f = 2K/\tau$. It has been shown that the signal can be perfectly sampled by *sinc* at rate ρ though $f(x)$ is obviously of non bandilmited response [7, 8]. It provides a novel point of view to interpret sampling of signals as a process to extract information out of the signal and to reconstruct it accordingly. It suggests a possible solution to the minimum resampling rate required for consistent resampling theory.

Unfortunately, there are two main obstacles to apply the innovation sampling theory to consistent resampling theory. First, the RI of a non trivial signal that belongs to $\mathcal{H}$ may not be straightly available. Second, the property of a "proper" acquisition filter is left undiscussed. In this paper, we tackle these two problems. We establish an upper bound of the RI for the signals in the general Hilbert space. We explore the minimum sampling rate above which the signal can be sampled in relation to an arbitrarily chosen acquisition filter. The results obtained herein are used to obtain the minimum resampling rate for consistent resampling, as present in Section 3.2. We use the result to examine the image processing and address the issue on how a lossless process can be designed.

2. RI of Signals in Shift Invariant Spaces

The rate of innovation ρ of a signal measures the degree of freedom of a signal per unit time. The degree of freedom, on the other hand measures the number of parameters required to uniquely specify the signal. For example, an N-th order polynomial defined by

$$f(x) = \sum_{i=0}^{N} c[i] x^i \tag{5}$$

is uniquely determined by the $N+1$ coefficients $c[i]$ and therefore the degree of freedom is $N+1$.

The RI is a function of the degree of freedom as well as its time span. It may be intuitive for some signals, e.g. the periodic pulse train (4). However, it is not so easy for other non-trivial signals. We observed that the RI of a signal depends on how much information of the signal is known *a priori*. For example, if we know that the signal in (5) crosses the axis at $x = a$, then $(x - a)$ is a factor of $f(x)$ and it can be expressed as

$$f(x) = (x - a) \sum_{i=0}^{N-1} c'[i] x^i \tag{6}$$

In this case, the degree of freedom of $f(x)$ is N and the time span of $f(x)$ is unchanged. In the frame of resampling where the signals are in the form of (2) and (3), an upper bound can be derived for the RI of such signals:

Proposition 2.1. *For a signal $f(x)$ of the form*

$$f(x) = \sum_k c[k] \phi\left(\frac{x}{T} - k\right) \tag{7}$$

where $\phi \in \mathcal{H}$ and T is a constant. Its rate of innovation ρ_f satisfies

$$\rho_f \leq \frac{1}{T} \tag{8}$$

Equality holds only when $\{\phi\left(\frac{x}{T} - k\right)\}_{k \in \mathbb{Z}}$ forms a Riesz basis.

Proof. For a signals given by (7), in every time interval $[kT, (k+1)T)$, there is one coefficient $c[k]$ to specify. This means that there is at most one degree of freedom every T seconds. If the coefficients are independent of each other, then $\rho_f = 1/T$. If the coefficients are not independent, then $\rho_f < \frac{1}{T}$.

To prove the condition for equality, we shall show that the coefficients $c[k]$ are independent if $\{\phi\left(\frac{x}{T} - k\right)\}_{k \in \mathbb{Z}}$ is a Riesz basis. Assume that the value of the coefficient $c[k_1]$ for a certain constant k_1 has been changed to $\Delta c[k_1]$. Using the set of sample values $\{\cdots, c[k_1 - 1], \Delta c[k_1], c[k_1 + 1], \cdots\}$ we can reconstruct a signal $\widetilde{f}(x)$ by

$$\widetilde{f}(x) = \sum_{k \neq k_1} c[k] \phi\left(\frac{x}{T} - k\right) + \Delta c[k_1] \phi\left(\frac{x}{T} - k_1\right) \tag{9}$$

To sample $\widetilde{f}(x)$ using the dual function ϕ_d of ϕ with a sampling period of T, the samples are given by:

$$\begin{aligned} \widetilde{f}[m]_T &= \left\langle \widetilde{f}(x), \phi_d\left(\frac{x}{T} - m\right)\right\rangle \\ &= \sum_{k \neq k_1} c[k] \left\langle \phi\left(\frac{x}{a} - k\right), \phi_d\left(\frac{x}{a} - m\right)\right\rangle \\ &+ \Delta c[k_1] \left\langle \phi\left(\frac{x}{a} - k_1\right), \phi\left(\frac{x}{a} - m\right)\right\rangle \end{aligned} \tag{10}$$

Since ϕ and ϕ_d are dual functions, they satisfy [4]

$$\left\langle \phi\left(\frac{x}{T}-k\right), \phi_d\left(\frac{x}{T}-m\right)\right\rangle = \delta[m-k] \quad (11)$$

Therefore, (10) is reduced to

$$\widetilde{f}_T[m] = \begin{cases} c[k], & m = k \neq k_1 \\ \Delta c[k_1], & m = k_1 \end{cases} \quad (12)$$

Thus a change in the value of $c[k_1]$ has no effect on the other samples. Hence we can conclude that the coefficients $c[k]$ are independent of each other. Further, since the dual function in (10) only exists when $\{\phi\left(\frac{x}{T}-k\right)\}_{k\in\mathbb{Z}}$ is a Riesz basis, the equality part of the Proposition 2.1 is proved. □

Let $\phi\left(\frac{x}{T}-k\right)$ be denoted by $\phi_{Tk}(x)$ for any $k \in \mathbb{Z}$. It can be expressed in the form of (7) as

$$\phi_{Tk}(x) = \phi\left(\frac{x}{T}-k\right) = \sum_m \delta[m-k]\phi\left(\frac{x}{T}-m\right) \quad (13)$$

The unit impulse sequences $\{\delta[m-k]\}_{m\in\mathbb{Z}}$ are independent of each other regardless of whether $\{\phi\left(\frac{x}{T}-k\right)\}_{k\in\mathbb{Z}}$ forms a Riesz basis. The RI of ϕ_{T_k} depends only on the sampling interval and equals to $\frac{1}{T}$.

If a signal $f(x)$ can be expressed as (7), then it can be viewed as weighted combinations of ϕ_{Tk}. Based on Proposition 2.1, we can say that the RI of such a signal cannot be greater than the RI of its component functions. Thus,

$$\rho_f \leq \rho_{\phi_T} = \frac{1}{T} \quad (14)$$

3. Acquisition Functions for Innovation Sampling

The principle behind having ρ_f as the minimum sampling rate is that N independent equations is required to solve for N unknowns uniquely. Hence, if $f(x)$ has an RI of ρ_f, then we need to solve for ρ_f unknowns per unit time to reconstruct $f(x)$. Each sample obtained in the interval $1/\rho_f$ is able to provide us with one equation, and a total number of ρ_f equations are needed per unit time.

To sample a signal at at its RI is possible only if a proper acquisition function is used. For example, consider a bandlimited signal $f(x)$ with a bandwidth of Ω_0. It can be expressed in the form of (7) with $\phi(x) = sinc(x)$ for all $1/T \geq \rho_f = \Omega_0/2\pi$. Although the bandwidth, or equivalently ρ_f and hence the minimum sampling rate, is a constant, the actual sampling rate $f_s = 1/T$ used to obtain the samples $f_T[k] = f(x)|_{x=kT}$ can be any value higher than the minimum. Thus the same $f(x)$ can be expressed as (7) using ϕ with different dilations. A possible interpretation is that when ϕ is dilated by $1/T$ and $f(x)$ is sampled at rate $f_s = 1/T$, each sample $c[k]$ obtained within an interval of T gives us a unique ϕ_{T_k}. From the various ϕ_{T_k} obtained, ϕ can be derived and $f(x)$ can be reconstructed.

In general, if a signal $f(x)$ can be expressed as (7), then a suitable acquisition function is the dual function of ϕ. If the dual ϕ_d exists, then the samples can be obtained by

$$c[k] = \left\langle f(x), \phi_d\left(\frac{x}{T}-k\right)\right\rangle \quad (15)$$

at sampling rate of $f_s = 1/T$. Proposition 2.1 tells us that $\rho_f \leq 1/T$ and the equality holds only when $\{\phi_{Tk}\}_{k\in\mathbb{Z}}$ is a Riesz basis. Therefore, the minimum sampling rate is attainable only when $\phi(x)$ is a generating function.

3.1. Sampling with General Acquisition Functions

However, it is very restrictive to require the acquisition or the resampling filter to be dual of the interpolation function. It is therefore desirable to derive the conditions on the set of acquisition functions such that a signal with finite rate of innovation of the form (7) can be sampled at its rate of innovation.

Here we restrict $\psi \in \mathcal{H}$ to generating functions and its dual function ψ_d exists. When $f(x)$ is sampled using ψ at ρ_f, the samples are given by

$$f[k] = \left\langle f(x), \psi\left[\rho_f\left(x - \frac{k}{\rho_f}\right)\right]\right\rangle \quad (16)$$

When ψ is known and ψ_d exists, $f(x)$ can be reconstructed from the samples $f[k]$ by

$$\widetilde{f}(x) = \sum_k f[k]\psi_d\left[\rho_f\left(x - \frac{k}{\rho_f}\right)\right] \quad (17)$$

Therefore, to derive the conditions on ψ is equivalent to examine the conditions on ψ_d such that a signal given by (7) can be equivalently represented by (17).

Our approach is to express both ϕ and ψ_d as polynomials. The Weierstrass's Approximation Theorem [4] states that every finite signal $\phi \in [a, b]$ where $a, b \in \mathbb{R}$ can be approximated arbitrarily well by a polynomial

$$P(x) = \sum_{k=0}^{n} c_1[k]x^k \quad (18)$$

such that

$$\|\phi - P\|_\infty \leq \epsilon \quad (19)$$

The order of P depends on ϵ, ϕ and the interval $[a, b]$. A reasonably approximation of P is given by the approximation order of that function. More specifically, if ϕ is a Maximum approximation Order with Minimum Support (MOMS) function of order L_ϕ, it can be expressed by weighted sum of derivatives of B-splines [5, 6]. Thus,

$$\phi(x) = \sum_{k=0}^{L_\phi - 1} p_k \frac{d^k}{dx^k} \beta^{L_\phi - 1}(x) \tag{20}$$

Since β^k is indeed a k-th order polynomial, (20) can be rewritten as

$$\phi(x) = \sum_{k=0}^{L_\phi - 1} c_1[k] x^k \tag{21}$$

Similarly, if ψ_d is an MOMS of approximation order L_{ψ_d}, it can be expressed in polynomial form as

$$\psi_d(x) = \sum_{k=0}^{L_{\psi_d} - 1} c_2[k] x^k \tag{22}$$

The number of coefficients associated with ψ_d is given by L_{ψ_d}.

The function $\phi_{T k}$ as defined in (13) can also be represented in the polynomial form by

$$\phi_{T k} = \sum_{m=0}^{L_\phi - 1} c_1[m] \left(\frac{x - kT}{T} \right)^m \tag{23}$$

When ϕ is known, for every k, in an interval of T, there is one $\phi_{T k}$ and its degree of freedom is L_ϕ. By substituting (23) into (7), we have

$$\begin{aligned} f(x) &= \sum_k c[k] \phi \left(\frac{x}{T} - k \right) \\ &= \sum_k \sum_{m=0}^{L_\phi - 1} (c[k] c_1[m]) \left(\frac{x - kT}{T} \right)^m \end{aligned} \tag{24}$$

Therefore, if $f(x)$ is represented in the polynomial form, the number of coefficients per unit time is at most L_ϕ / T. Similarly, substituting (22) into (17), we have

$$\widetilde{f}(x) = \sum_k \sum_{m=0}^{L_{\psi_d} - 1} (f[k] c_2[m]) (\rho_f x - k)^m \tag{25}$$

For every sample $f[k]$ in an interval of $1/\rho_f$, there are at most L_{ψ_d} coefficients. Comparing (24) and (25), the conditions on ψ_d such that $f(x)$ can be represented by $\widetilde{f}(x)$ can be obtained.

Proposition 3.1. *Let $f(x)$ be a signal given by (7) with an RI of ρ_f. Assume that ϕ is a MOMS function with an approximation order of L_ϕ. $f(x)$ can be represented by another MOMS function $\psi_d \neq \phi$ with approximation order L_{ψ_d} at dilation level ρ_f as given by (17) if*

1. *$L_{\psi_d} \geq L_\phi$; and*
2. *the set $\{\psi_d(x - \frac{k}{\rho_f})\}_{k \in \mathbb{Z}}$ is Riesz basis of its span.*

Proof. From (24), $f(x)$ is a polynomial of order $L_\phi - 1$. On the other hand, from (25), the order of polynomials that can be represented by ψ_d is $L_{\psi_d} - 1$. Therefore, if $f(x)$ can be represented by ψ_d as in (17), then

$$L_{\psi_d} \geq L_\phi \tag{26}$$

When $f(x)$ is represented by ψ_d at dilation level ρ_f as in (17), the RI of $f(x)$ is equal to the inverse of the dilation level. Following Proposition 2.1, equality holds only when $\{\psi_d(x - \frac{k}{\rho_f})\}_{k \in \mathbb{Z}}$ is Riesz basis of its span. Hence this proposition is proved. □

It can be shown that the approximation order of ψ is identical to the approximation order of its dual ψ_d.

Proposition 3.2. *The approximation orders of a function ψ and its dual ψ_d are the same.*

Proof. If the approximation order of ψ is L_ψ, its frequency response $\Psi(\Omega)$ satisfies the Strang-Fix condition given by

$$\begin{cases} \Psi[2\pi k] = \delta[k] \\ \Psi^{(m)}[2k\pi] = 0, \quad k \in \mathbb{Z}, \ m = 0, \cdots, L_\varphi - 1 \end{cases} \tag{27}$$

We shall use mathematical induction to prove that the frequency response of ψ_d satisfies the Strang-Fix condition as well.

The frequency response of φ_d can be specified as [9]:

$$\Psi_d(\Omega) = \frac{\Psi(\Omega)}{A_\psi(\Omega)} \tag{28}$$

where

$$A_\psi(\Omega) = \sum_k |\Psi(\Omega + 2k\pi)|^2 \tag{29}$$

The first order derivative of $\Psi_d(\Omega)$ with respect to Ω is given by

$$\begin{aligned} \Psi_d^{(1)}(\Omega) &= \frac{d}{d\Omega} \Psi_d(\Omega) = \frac{d}{d\Omega} \frac{\Psi(\Omega)}{A_\psi(\Omega)} \\ &= \frac{\Psi^{(1)}(\Omega) A_\psi(\Omega) - \Psi(\Omega) A_\psi^{(1)}(\Omega)}{A_\psi^2(\Omega)} \end{aligned} \tag{30}$$

where

$$A_{\psi}^{(1)}(\Omega) = \frac{d}{d\Omega}A_{\psi}(\Omega) = 2\sum_{k}|\Psi(\Omega+2k\pi)|\,|\Psi^{(1)}(\Omega+2k\pi)| \quad (31)$$

For $\Omega = 2n\pi$, $n \in \mathbb{Z}$,

$$A_{\psi}^{(1)}(\Omega)|_{\Omega=2n\pi} = 2\sum_{k}|\Psi\left[2(k+n)\pi\right]|\,|\Psi^{(1)}\left[2(k+n)\pi\right]| = 0 \quad (32)$$

since $\Psi^{(1)}\left[2(k+n)\right]\pi = 0$ for all $k, n \in \mathbb{Z}$. Substituting (32) into (30), we have

$$\Psi_{d}^{(1)}(\Omega)|_{\Omega=2n\pi} = \frac{\Psi^{(1)}(\Omega)A_{\psi}(\Omega) - \Psi(\Omega)A_{\varphi}^{(1)}(\Omega)}{A_{\psi}^{2}(\Omega)} = \frac{0-0}{A_{\psi}^{2}(\Omega)} = 0 \quad (33)$$

Assume that $\Psi_{d}^{(m-1)}(\Omega)$ satisfies the Strang-Fix condition. Then for $n \in \mathbb{Z}$ and $m = 0, \cdots, L_{\psi}-1$, we have

$$\Psi_{d}^{(m-1)}(2n\pi) = 0 \quad (34)$$

From the chain rule of differentiation,

$$\Psi_{d}^{(m)}(\Omega) = \frac{d^m}{d\Omega^m}\Psi_{d}(\Omega) = \frac{d}{d\Omega}\Psi_{d}^{(m-1)}(\Omega) = \Psi_{d}^{(m)}(\Omega)\Psi_{d}^{(1)}(\Omega) \quad (35)$$

Hence,

$$\Psi_{d}^{(m)}(\Omega)|_{\Omega=2n\pi} = \Psi_{d}^{(m)}(2n\pi)\Psi_{d}^{(1)}(2n\pi) \quad (36)$$

Since $\Psi_{d}^{(1)}(2n\pi) = 0$, we have $\Psi_{d}^{(m)}(2n\pi) = 0$ for $n \in \mathbb{Z}$ and the Strang-Fix condition is satisfied for $\Psi_{d}^{(m)}(\Omega)$. By mathematical induction, we have $\Psi_{d}^{(m)}(2n\pi) = 0$ for $n \in \mathbb{Z}$, $m = 0, \cdots, L_{\varphi}-1$. Therefore, $\Psi_{d}(\Omega)$ satisfies the Strang-Fix condition up to order $L_{\psi}-1$ and the approximation order of ψ_d is L_{ψ} as well. □

Therefore, to choose a function ψ such that a signal in the form of (7) can be sampled at its RI, from Proposition 3.2 and Proposition 3.1, it requires the approximation order of ψ satisfying $L_{\psi} \geq L_{\phi}$ and ψ is a generating function.

Another possible interpretation of Proposition 3.1 is as follows. When $f(x)$ is expressed in the form of (24), within each interval $[kT, (k+1)T)$, the coefficients $\{c_1[m]\}_{m\in[0,L_{\phi}-1]}$ are known. For each sample $c[k]$ obtained, the total number of coefficients is given by $N_1 = L_{\phi}$. Therefore, it has N_1 degrees of freedom per unit time. Similarly, for a signal given by (25), within each interval $[k/\rho_f, (k+1)/\rho_f)$, for each ψ_{dT_k} the number of parameters is $N_2 = L_{\psi_d}$. In order to represent $f(x)$ using ψ_d, we require

$$\frac{N_2}{1/\rho_f} \geq \frac{N_1}{T} \Rightarrow L_{\psi_d} \geq \frac{L_{\phi}}{T\rho_f} \quad (37)$$

From Proposition 2.1, $\rho_f \leq \frac{1}{T}$ and therefore $L_{\psi_d} \geq L_{\phi}$.

This interpretation can lead to a more general condition on the dilation level D for an arbitrary acquisition function φ such that $f(x)$ can be represented by

$$f(x) = \sum_{k} f_D[k]\varphi\left[\frac{1}{D}(x-kD)\right] \quad (38)$$

Proposition 3.3. *Let ϕ be a known function and $f(x)$ be a signal given by (7) with a finite RI of ρ_f. If $f(x)$ is to be expressed using a function φ as in (38), then the dilation level D should satisfy*

$$D \leq \begin{cases} T\frac{L_{\varphi}}{L_{\phi}}, & L_{\phi} \geq L_{\varphi} \\ T, & L_{\phi} \leq L_{\varphi} \end{cases} \quad (39)$$

where both ϕ and φ are assumed to be MOMS functions with approximation orders L_{ϕ} and L_{φ} respectively.

Proof. Given $f(x)$ as in (38), it can be sampled by using the acquisition function φ which is the dual function of φ at the rate $1/D$. The sample values are

$$f[k] = \left\langle f(x), \varphi\left[\frac{1}{D}(x-kD)\right]\right\rangle \quad (40)$$

$$= \left\langle f(x-kD), \varphi\left(\frac{x}{D}\right)\right\rangle \quad (41)$$

For every k, we have a $\varphi_{D_k} = \varphi(\frac{x}{D}-k)$ similar to (**??**). If the approximation order of φ is L_{φ}, then the degree of freedom is L_{φ}.

On the other hand, if $f(x)$ is expressed as a polynomial of order $L_{\phi}-1$ as in (24), then the degree of freedom is L_{ϕ}. Thus for each k, the degree of freedom must be given by

$$N_1 = \min(L_{\phi}, L_{\varphi}) \quad (42)$$

From Proposition 3.2, $L_{\varphi} = L_{\varphi_d}$. Therefore

$$N_1 = \min(L_{\phi}, L_{\varphi_d}) \quad (43)$$

Following Proposition 3.1, the degree of freedom per unit time is L_{ϕ}/T. To express $f(x)$ using φ_d of dilation level $1/D$, it requires

$$\frac{N_1}{D} \geq T\frac{L_{\phi}}{T} \quad (44)$$

If $L_\phi \geq L_\varphi$, then $N_1 = L_{\varphi_d}$ and hence

$$D \leq T\frac{L_{\varphi_d}}{L_\phi} \tag{45}$$

If $L_\phi \leq L_{\varphi_d}$, then $N_1 = L_\phi$ and

$$D \leq T \tag{46}$$

Given an acquisition function φ of approximation order L_φ, the dilation level and hence the corresponding sampling rate can be chosen directly by using (39). □

3.2. Application to Consistent Resampling Theory

In a resampling system, the input sequence $f_T[n]$ is interpolated by the interpolation function ϕ to produce $\widetilde{f}_1(x)$ as in (2). The output $f_{T'}[m]$ is obtained by resampling $\widetilde{f}_1(x)$ at the rate $1/T'$.

$$f_{T'}[m] = \left\langle \widetilde{f}(x), \psi\left(\frac{x}{T'} - m\right)\right\rangle \tag{47}$$

Assume that ϕ and ψ are MOMS functions with approximation orders L_ϕ and L_ψ respectively. In order to produce $\widetilde{f}_2$ using $f_{T'}[m]$ such that $\widetilde{f}_1 = \widetilde{f}_2$, from Proposition 3.3, the resampling interval T' is required to satisfy (39).

The results obtained can be applied to image processing. In general, the original image is assumed to be ideally sampled by the impulse train with sampling period $T = 1$. The commonly used interpolation functions include the finitely defined B-splines and the resampling function is $\psi = \delta$. From [6], B-spline functions are MOMS functions. The approximation order of a B-spline of order $n-1$ is $L = n$. For example when bilinear function is used, $\phi = \beta^1$ and $L_\phi = 2$. On the other hand, since the dual of $\psi = \delta$ is $\psi_d = sincx$, from Proposition 3.2, the approximation order of ψ is given by $L_\psi = L_{\psi_d} = 1$. Since $L_\psi < L_\phi$, according to (39), the resampling period should satisfy

$$T' \leq T\frac{L_\psi}{L_\phi} \Rightarrow T' \leq \frac{1}{2} \tag{48}$$

in order to resample the original image consistently.

Therefore, in order to process an image losslessly using consistently resampling with β^1 and δ as the interpolation and resampling function, the resampling rate should be at least twice of the original sampling rate. It implies that the zoom in operation of factors less than 2 and any zoom out operation would inevitably cause information loss. In order to pursue lossless zoom out, functions of higher approximation order should be used as the resampling function, such as higher order B-splines.

4. CONCLUSIONS

In this paper we provided an upper bound on the RI for signals in shift invariant spaces. We also specified the criteria for choosing a proper acquisition function for innovation sampling. Based on these results, a lower bound on the resampling rate used in consistent resampling is developed. This bound can be used to decide the globally minimum sampling rate to achieve lossless resampling of a signal without bandlimited constraint.

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