Demodulation of UWB Impulse Radio Signals Using B-splines

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***Abstract*— We propose a novel approach to demodulate time-hopping pulse position modulated (PPM) impulse radio signals using a fully digital receiver. The pulses are viewed as non-uniform samples of an underlying continuous signal. They are then interpolated and *consistently* uniformly resampled using an appropriate kernel. A non-uniformly sampled signal is *consistently resampled* if the original samples can be deduced from the resampled uniform sequence using a suitable methodology. Such resampled sequence can be used to determine the position of the original pulses. In this paper, we showed how the B-splines can be used as an interpolating kernel and an algorithm to detect the position of the PPM pulses is developed.**

I. Introduction

Time-hopping impulse radio has been proposed as a simple ultra-wideband (UWB) wireless communication technique. It transmits a stream of pulse-position modulated (PPM) pulses that are of very short (sub-nanosecond) duration [1], [2]. The time-hopped PPM signal using a basic pulse $p(t)$ can be mathematically expressed as [3]

$$s(t) = \sum_j p(t - jT_s - c_j T_c - a_j \epsilon) \tag{1}$$

where T_s is the symbol rate and T_c is the chip rate. Demodulating such signals involves estimating the time location of each pulse. We can simplify (1) by using a single variable t_j to represent the time shift, giving us

$$s(t) = \sum_j p(t - t_j \epsilon) \tag{2}$$

Since the duration of the pulses are very short and strictly non-overlapping, the signal $s(t)$ can be assumed to be a sequence of Diracs $x(t)$ where

$$x(t) = \sum_{k \in \mathcal{Z}} c_k \delta(t - t_k) \tag{3}$$

Many methods have been proposed to design the pulse shapes and to demodulate the signal. The most common demodulation scheme is matched detection based on the maximum likelihood criterion [3]. It is implemented in the analog domain. However, a fully digital receiver is more desirable as it will provide more flexibility. Designing a digital receiver for such a wide bandwidth signal is not simple. The main problem is the high sampling rate required since according to Shannon's sampling theorem, the sampling frequency should be at least twice the bandwidth of the signal to avoid aliasing. For signal with bandwidth in the range of several gigahertz used in UWB systems, the minimum sampling rate will be extremely daunting. Some methods use N-branch filter banks such that each branch only need to sample at $1/N$ of the Nyquist rate. However, this approach will increase the hardware implementation cost.

In [4], a new point of view on sampling is presented which is applicable to signals that are not necessarily bandlimited. It will be referred to as *innovation sampling*. The rate of innovation is defined to be the number of unknowns per unit time. The innovation sampling theory states that signals with finite rate of innovation can be perfectly reconstructed if sampled uniformly at a rate that at least equals their rate of innovation using an appropriate sampling kernel. For a stream of Diracs, the rate of innovation is related to the number of pulses per unit time. This implies that the required sampling rate is much lower than the Nyquist rate. However, as we will discuss in Section II, the reconstruction algorithm for innovation sampling theory is more complicated than Shannon's uniform sampling theorem and often involves root finding. If this theory is to be practically applied to the demodulation of impulse radio signals, then a more efficient algorithm is needed.

In this paper, we develop a more efficient algorithm to reconstruct and hence demodulate the PPM signals by reinterpreting the innovation sampling theory in terms of resampling. The reconstruction can be decomposed into a resampling process, followed by an algorithm to reconstruct the pulses. The interpolating kernel used in the resampling stage needs to be chosen properly so that the information contained in the pulses can be preserved. The term *consistent resampling* is therefore introduced to define a resampling process which has this property. This theory is presented in Section II. Algorithms based on B-splines to reconstruct the pulses in the continuous as well as the discrete domain are developed in Sections III and IV respectively. We also showed that our algorithm outperforms the one proposed in [4] in terms of computational complexity. In Section V we show by examples how the algorithm works and we conclude the paper in Section VI.

II. Innovation Sampling Theory Revisited

The innovation sampling theory shows that the signals with finite rate of innovation can be sampled uniformly and perfectly reconstructed using appropriate sampling kernels. Consider the periodic stream of pulses,

$$x'(t) = \sum_{k=0}^{K-1} c_k \sum_{n} \delta\left(t - t_k - n\tau\right) \tag{4}$$

The rate of innovation is $\rho = 2K/\tau$. Let the sequence $x'[k]$ be $x'[k] = f'(t)\delta(t - t_k)$, $f'(t)$ is an arbitrary continuous signal where $c_k = f'(t)|_{t=t_k}$, therefore $x'(t) = x'[k]$ mathematically. According to the innovation sampling theory, $x'[k]$ is first interpolated using $h_B(t) = Bsinc(Bt)$ where $B \geq \rho$

$$y(t) = x'[k] * h_B(t) = \sum_{k,n} x'[k] h_B(t - t_k - n\tau) \tag{5}$$

Then the interpolated signal in (5) is resampled uniformly

$$y_T[m] = y(t) \cdot \delta(t - mT) = \sum_{k,n} x'[k] h_B(mT - t_k - n\tau) \tag{6}$$

The subscript T is used to indicate the uniform sampling rate of the sequence. The innovation sampling theory provides us an algorithm to reconstruct $x'[k]$ from the resampled sequence $y_T[m]$. In this Section, we interpret the innovation sampling theory in the resampling system and draw some guidelines to develop our own algorithm.

A. Consistent Resampling

In order to reconstruct a sequence from its resampled sequence, the resampling process should be reversible. We define two sequence are *consistent* if and only if they are be resampled to one another. In other words, for two sequences $f[m]$ and $g[n]$ of length M and N respectively, where $f[m] = c_m\delta(t - t_m)$ and $g[n] = d_n\delta(t - t_n)$. Giving an interpolating kernel $\phi(t)$, if

$$y(t) = g[n] * \phi(t) = f[m] * \phi(t) \text{ and} \tag{7}$$

$$\forall m, c_m = y(t)|_{t=t_m} \qquad \forall n, d_n = y(t)|_{t=t_n} \tag{8}$$

The two sequences are said to be consistent subject to $\phi(t)$. It may be counter-intuitive that a sequence can be represented by a shorter sequence completely. As it can be observed from the definition of consistency, it is subject to a certain interpolator and the information can be stored in the interpolator as well. The two sequences are two different representations of the same underlying signal $y(t)$. Assume that $f[m]$ is the input to the resampling system and $g[n]$ is the output, the signal information of $f[m]$ is stored in $y(t)$, and transferred to $g[n]$. Assume $g[n]$ is uniformly resampled at T, the output is given by

$$\begin{aligned}\sum_{n} g_T[n] &= \sum_{n,m} f[m] * \phi(t) \cdot \delta(t - nT) \\ &= \sum_{n,m} c_m\phi(nT - t_m) = \sum_{m} f[m] * \phi_T[n]\end{aligned} \tag{9}$$

Obviously, the resampling system has discrete operator $\phi_T[n]$. It is therefore essential to choose a suitable interpolating kernel for the input $f[m]$.

A possible interpretation of the consistent resampling can be found in the information theory [5]. Consider $f[m]$ and $g[n]$ as the discrete input and output of the transceiver represented by $\phi_T[n]$. The total information needed to reconstruct the pulse, is an analog to the entropy (the uncertainty) of the signal. As a result of Markov process, the information contained in $g_T[n]$ can only be as much as in $f[m]$ if and only if $\phi_T[n]$ is non singular. Therefore, for the resampling process to be reversible, we require $\phi_T[n]$ defined a non singular process, or $\phi_T[n] \neq 0$ for all n. Under this circumstances, there exists certain algorithm to reverse the resampling process and $f[m]$ can be reconstructed such that c_m, t_m are made available. It is noteworthy that the entropy depends on both the entropy rate (rate of innovation) and time (number of samples); consequently, we can always find a consistent resampled sequence for any resampling rate provided the output sequence is long enough.

B. Innovation sampling theory in resampling system

We restate the innovation sampling theory in frames of consistent resampling. From (6), interpolate $y_T[n]$ using $h_B(t)$ we have

$$\begin{aligned}&\sum_{m} y_T[m] * h_B(t) \\ &\quad = \sum_{m,k,n} x'[k] h_B(mT - t_k - n\tau) h_B(t - nT) \\ &\quad = \sum_{k} x'[k] \sum_{n} h_B(t - t_k - n\tau) \\ &\quad = \sum_{k} x'[k] * h_B(t)\end{aligned} \tag{10}$$

since $h_B[(m-n)T] = \delta_{m,n}$ where $\delta_{m,n}$ defines a Kronecker Delta. The condition in (7) is satisfied. By substituting $t = t_k$ into (5), the condition stated in (8) is satisfied as well. Therefore, $x'(t)$ and $y_T[m]$ are consistent.

One not so obvious point in the innovation sampling theory is that it requires a minimum number of samples to recover the pulses, instead of the lowest sampling rate that normally used in sampling theory. To understand this problem, we check how the information of the input sequence is reserved. For a pulse $\delta(t-t_k)$, the frequency response is $\delta(t-t_k) \overset{FT}{\to} e^{-i2\pi t_k f}$. Due to the linearity of fourier transform, the frequency response of the signal in (4)is given by

$$X'(f) = \sum_{k=0}^{K-1} \sum_{n\in\mathbb{Z}} c_k e^{-i2\pi t_k(f - nf_0)} \tag{11}$$

$X'(f)$ is a periodic signal of periodicity $f_0 = 1/\tau$. It can be reconstructed by any part of bandwidth f_0 of $X'(f)$. For the reason of simplicity, we pass the signal $X'(f)$ to a low pass filter of bandwidth $f_0/2$ and denote it by $X_0(f)$.

Take an inverse Fourier transform of $X_0(f)$, we have the aliased signal $x_0(t)$, which has highest frequency component

$f_0/2$. According to Shannon's theory, $x_0(t)$ can be fully characterized by its samples of Nyquist rate $f \geq 2 \cdot \frac{f_0}{2} = 1/\tau$. Despite the actual choice of sampling rate, the rate of innovation of x_0 is nevertheless a constant $\rho_{x_0} = 1/T = \tau$. To recover $x'(t)$, the information we need is $I_{x'} = 2K$; therefore, the time needed to transmit the total amount is at least $t = I_{x'}/\rho_{x_0} = 2K/\tau$ which corresponding to $N = t/T = 2K$ number of pulses. The number of samples required is consistent with the result in [4].

The next task is to reconstruct the input sequence from its resampled output. To reconstruct a sequence of K pulses, it needs to solve a Kth order Yule-Walker system for an annihilating filter $A(z)$, followed by factoring $A(z)$ to find the position of the pulses. Another Vandermonde system is solved for the weights of the pulses. The procedure is very complex and the computational complexity depends on the efficiency of several other algorithms.

Based on the understanding of consistent resampling, we develop an algorithm using B-spline as the interpolating kernel in Section III and Section IV. As it will become clear later, our algorithm has a reduced complexity comparing to the innovation sampling theory.

III. Sampling with continuous B-splines

A B-spline of order n, $t \in R$ is given by

$$\beta^n(t) = \sum_{j=0}^{n+1} \frac{(-1)^j}{n!} \binom{n+1}{j} \cdot \left(t + \frac{n+1}{2} - j\right)^n \mu\left(t + \frac{n+1}{2} - j\right) \quad (12)$$

where $\mu(x) = 1$ only if $x \geq 0$. Assume $n > 1$, the first derivative of B-spline is given by [6]

$$\beta^n(t)' = \frac{d\beta^n(t)}{dt} = \beta^{n-1}\left(t + \frac{1}{2}\right) - \beta^{n-1}\left(t - \frac{1}{2}\right) \quad (13)$$

If we interpolate a stream of pulses as in (3) using $h(t) = \beta^n(t)$, we have

$$\widehat{f}(t) = \sum_k c_k \beta^n(t - t_k) \quad (14)$$

Substituting (12) and (13) and differentiating (14) gives

$$\frac{d\widehat{f}(t)}{dt} = \sum_k c_k \left[\beta^{n-1}\left(t + \frac{1}{2} - t_k\right) - \beta^{n-1}\left(llt - \frac{1}{2} - t_k\right)\right] \quad (15)$$

The continuous signal $\widehat{f}(t)$ contains all the information we need to reconstruct c_k and t_k.

Proposition 1: Consider a stream of pulses $x(t) \in L^2(\mathbb{R})$ as given in (3). Assume that the coefficient of the pulses c_k are independent of the locations t_k. Let the sampling kernel be $\varphi(t) = \beta^n(t), n > 1$, the resultant signal $\widehat{f}(t) = x(t) * \beta^n(t)$ is sufficient to localize the pulses using differentiation.

proof: Let the set $t = t_c$ contain all the roots that $\frac{d\widehat{f}(t)}{dt}|_{t_c} = 0$. The proposition can be proved in two steps. First, we show that all t_k can be found in t_c. Second, all t_c are related to t_k in such a way that it is possible to eliminate the t_c other than t_k.

Step 1: for $t = t_k$, since c_k are independent of t_k, to differentiate $\widehat{f}(t)$ is equivalent to differentiate the underlying β^n. Therefore, (15) becomes

$$\frac{d\widehat{f}(t)}{dt} = \sum_k c_k[\beta^{n-1}(\frac{1}{2}) - \beta^{n-1}(-\frac{1}{2})] \quad (16)$$

The B-spline is symmetric and $\beta^n(t) = \beta^n(-t)$. Hence, $d\widehat{f}(t)/dt = 0$ and thus $t_k \in t_c$.

Step 2: since B-spline is locally defined and differentiable at the end points, other possible solution occurs when

$$\beta^{n-1}(t_c - t_k + 1/2) - \beta^{n-1}(t_c - t_k - 1/2) = 0 \quad (17)$$

The n^{th} order spline has support $|t| \leq (n+1)/2$. Also, $\beta^n(t) = 0$ when $|t| = (n+1)/2$. Solving (17), we get

$$t_k - \frac{n+1}{2} \geq t_c \geq t_k + \frac{n+1}{2} \quad (18)$$

In addition, for $f(t_c)$ to be valid,

$$t_k - \frac{n+1}{2} \leq t_c \leq t_k + \frac{n+1}{2} \quad (19)$$

Therefore, we can conclude that in general

$$t_c = t_k \pm (n+1)/2 \quad (20)$$

The third possible set of t_k occurs when

$$\sum_k c_k \beta^{n-1}\left(t + \frac{1}{2} - t_k\right) = \sum_k c_k \beta^{n-1}\left(t - \frac{1}{2} - t_k\right) \quad (21)$$

which has the same solution set as in (20). Therefore, we conclude that for all t_c that $\frac{d\widehat{f}(t)}{dt}|_{t_c} = 0$, they are related to t_k by (1) the t_k itself, (2) $t_c = t_k \pm (n+1)/2$. Therefore, it is possible to work out t_k from the t_c. An algorithm shown in Table I is designed as one possible way to do so. Once t_k are available, the coefficients can be obtained by $c_k = \widehat{f}(t)|_{t=t_k}$ and the pulses are reconstructed.

```
input: roots found m[]
output: t_k, k[], flag[];
initialize flag[]=0;
for i=1; i++
      for j=i+1, j++
            if m[j]-m[i]=2
                  flag[i] = (1+flag[i])mod 2;
                  flag[j] = (1+flag[j])mod 2;
            else if m[j]-m[i] < 2 break;
            else m[j]-m[i]> 2 move to next i;
p=1;
for z=1, z++
      if f[z] == 0 & f(m[z]) ≠ 0
            k[p] = m[z]; p++
```

TABLE I

Algorithm to separate t_k from its variants $t_k \pm (n+1)/2$

Since B-splines are of local support, for any point on $\widehat{f}(t)$, only pulses have distances less than $(n+1)/2$ are required to determine the point. The B-splines can be adjusted by a factor m such that $\beta_m^n(t) = \beta^n(mt)$, the width of time spread is $L_m^n = (n+1)/m$. The memory requirement for the algorithm given in Table. I is $O\left(\frac{n+1}{m\cdot(t_k - t_{k-1})_{min}}\right)$ for all k. The complexity is linear with respect to the number of t_c, which is at most three times the number of the pulses. Comparing to the Innovation sampling theory, we find that our algorithm involves roots finding of the 1st order differentiation equation and a simple algorithm as in Table I, which is of complexity $O(t_c^2)$.

IV. Sampling with discrete B-splines

Differentiation is normally restricted to continuous signal. B-spline has a distinctive merit to bridge discrete and continuous domain such that this operation can be applied in discrete domain [6]. This property of B-spline enables us to carry out the differentiation process on digital processing machines.

Define the discrete B-spline $b_m^n(k)$

$$b_m^n(k) := \beta^n(k/m) \qquad |k| \leq m(n+1) \tag{22}$$

the B-spline coefficient $g[k]$ of a signal $f(t)$ can be obtained from direct transform

$$g[k] = (b_1^n)^{-1} * f[k] \quad f[k] = f(t)|_{t=k}, k \in \mathbb{Z} \tag{23}$$

Since the B-splines are not Nyquist function except for $n=1$, the $g[k] \neq f[k]$; the second condition of consistent sampling should be adjusted. Define the dual operator of $\beta^n(t)$ to be $\widetilde{\beta}^n(t)$, the second condition is restated as

$$\forall m, c_m = y(t) * \widetilde{\beta}^n(t - t_m) \qquad \forall n, d_n = y(t) * \widetilde{\beta}^n(t - t_n)$$

Proposition 2: Consider a sequence of pulses given by (3). Resample it at $t = nT, n = 0, 1, \cdots 2K$ using β^n as interpolating kernel. Then the regular samples $y_T[n] = \widehat{f}(t)\delta(t - nT)$ is sufficient to localize the pulses in $x(t)$ where $\widehat{f}$ is defined in (14).

We have proved that the continuous representation $\widehat{f}(t)$ contains sufficient information to extract t_k. Here, we only need to show that the regular samples $y_T[n]$ is differentiable and contains sufficient information to reconstruct t_k.

proof: From (14),

$$y_T[m] = \widehat{f}(t)\delta(t - nT) = \sum_{k=0}^{K-1} c_k \beta^n(mT - t_k) \tag{24}$$

Similar to the procedure used in continuous domain, we differentiate the discrete $y_T[n]$ using the differentiability of discrete B-splines. The sequence $y_T[n]$ is transformed to obtain its B-spline coefficient, followed by two discrete filters to obtain its differentiation [6]. The system is shown in Figure 1.

The shifted B-spline C_1^n is defined by $c_m^n(k) = \beta^n(k/m + 1/2)$. This ensures the z transform of $c_1^n(k)$ would have a common term $\frac{(1+z)}{2}$ [6]. We denote

$$C_1^n(z) = \frac{1+z}{2} C^n(z) \tag{25}$$

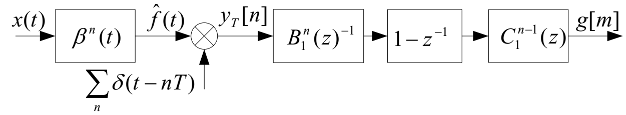

Fig. 1. Differentiate a discrete sequence using B-spline, the exact meaning of each symbol can be found in Section IV.

Let the z transform of $y_T[m]$ be $Y_T(z)$, following the steps as shown in Figure 1, we have:

$$Y_T(z) \xrightarrow{z} \sum_m \sum_k c_k \beta^n(mT - t_k) z^{-m} \tag{26}$$

$$(b_1^n)^{-1}(k) \xrightarrow{z} \frac{z^{[n/2]}}{b_1^n([n/2]) \prod_{i=1}^{[n/2]} (z - z_i)(z - z_i^{-1})} \tag{27}$$

where $\{(z_i, z_i^{-1})\}$ are roots of $B_1^n(z)$, the first differentiation result of the sequence $y_T[n]$ in B-spline is given by $g[m]$ where

$$\begin{aligned} G(z) &= Y_T(z) \cdot (1 - z^{-1}) \cdot C_1^{n-1}(z) \\ &= \frac{\sum_m \sum_k c_k \beta^n(t - mT - t_k) z^{-\left[\frac{n}{2}\right]}}{2b_1^2([n/2]) \prod_{i=1}^{[n/2]} (z - z_i)(z - z_i^{-1})} \\ &\quad \cdot (1 - z^{-1})(1 + z) C^{n-1}(z) \end{aligned} \tag{28}$$

Set $G(z) = 0$ and return to time domain, we have for

$$\begin{aligned} \forall m, &\sum_k c_k \beta^n \left[t - (m+1)T - t_k\right] \\ &= \sum_k c_k \beta^n \left[t - (m-1)T - t_k\right] \end{aligned} \tag{29}$$

Comparing the equation with (15), we confirm it can be solved using the same technique for (15). To conclude, the regular samples $y_T[n]$ contains all information needed to reconstruct $x(t)$.

V. Examples

We demonstrate with examples in both continuous and discrete domain. Since the major problem is to localize the pulses, we normalize all coefficients to 1.

Example 1: Consider a sequence of six pulses with no noise component as shown in Figure 2. This examples shows all possible positions between neighboring shifts of the underlying kernel $\beta^3(mt)$ with $m = 1$. Our algorithm in Table. I returns the exact position of t_k.

Example 2: The stream of pulses is resampled using interpolating kernel $\beta^2(t)$, the simplest B-spline which is first order differentiable. Let a sequence of pulses consist of eight distinctive pulses in a period of 128. The locations of the pulses are stated in Table II. The sequence is not restricted to periodic sequence. The sequence is resampled to $y_T[n], n = 0, 1 \cdots, N-1$ where $N = 2*8 = 16$ samples is collected. The z transform is calculated using FFT where $N = 128$ and set $z = e^{i\omega}$. Then the sequence is differentiate in B-spline domain following the steps (26) to (28). The results are stored in Table III. Passing the sequence of $g[m]$ through the algorithm in Figure I, and return the values of m. Our estimated answers are shown in the last row of Table II which are correct.

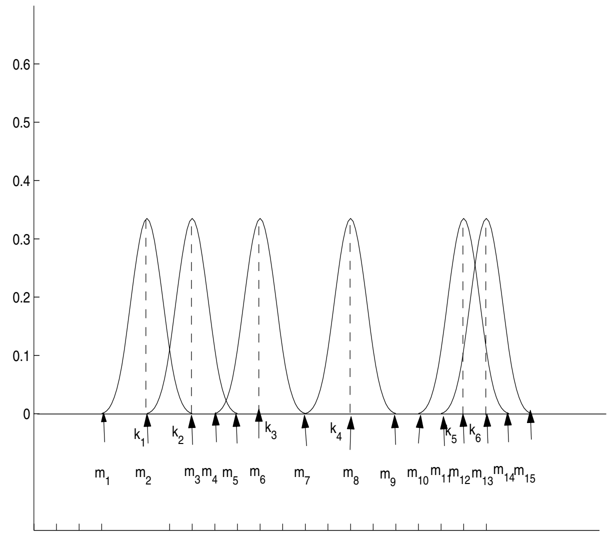

Fig. 2. Stream of Diracs of six pulses at k_1 to k_6. The roots of $f'(t)$ ranges from m_1 to m_{15}. The vertical axis denotes the amplitude of the pulses, which is trivial in this example. The horizontal axis denotes the arriving time of the pulses

TABLE II

THE DESIRED AND THE ESTIMATED VALUES OF t_k.

	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
original	9	15	19	32	57	62	82	105
estimated	9	15	19	32	57	62	82	105

VI. CONCLUSION

In this paper, we propose a novel method to demodulate the PPM pulses by treating it as a non-uniformly sampled sequence. We prove that the pulses can always be reconstructed from its consistent resampled sequence given the implicitly used interpolating kernel in resampling is not singular. Therefore the resampling process is reversible, and the information contained in the pulses can be reserved and passed to the resampled sequence. Our next task is to design an algorithm that can be used to extract the information out from the resampled sequences. We design an algorithm suitable for reconstructing pulses using B-spline in both continuous and discrete domain. We show that our algorithm not only demodulate the PPM pulses accurately, but outperform the innovation sampling theory in terms of computational complexity. Future works can be done to develop guidelines to find an interpolating kernel such that the number of samples needed to reconstruct the pulses is minimum, as well as an algorithm that is robust to noise.

TABLE III

VALUE OF $G(z)$. 0 AND x STAND FOR ZERO AND NONZERO VALUES OF $G(z)$ RESPECTIVELY.

m	$G(z)$							
$9-16$	0	x	x	x	0	0	0	x
$17-24$	x	x	0	x	x	x	0	0
$29-36$	0	0	0	0	x	x	x	0
$57-64$	0	x	x	x	0	0	x	x
$65-72$	x	0	0	0	0	0	0	0
$81-88$	0	0	x	x	x	0	0	0
$105-112$	0	x	x	x	0	0	0	0

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