Stability and Statistical Properties of Second-Order Bidirectional Associative Memory

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Abstract—In this paper, a bidirectional associative memory (BAM) model with second-order connections, namely second-order bidirectional associative memory (SOBAM), is first reviewed. The stability and statistical properties of the SOBAM are then examined. We use an example to illustrate that the stability of the SOBAM is not guaranteed. For this result, we cannot use the conventional energy approach to estimate its memory capacity. Thus, we develop the statistical dynamics of the SOBAM. Given that a small number of errors appear in the initial input, the dynamics shows how the number of errors varies during recall. We use the dynamics to estimate the memory capacity, the attraction basin, and the number of errors in the retrieved items. Extension of the results to higher-order bidirectional associative memory is also discussed.

Index Terms—Associative memory, BAM, neural network, stability.

I. INTRODUCTION

ASSOCIATIVE memories [1], [2] have been intensively studied in the past decade. An important feature of associative memories is the ability to recall the stored items from partial or noisy inputs. One form of associative memories is the bivalent additive bidirectional associative memory (BAM) [3]. It is a two-layer heteroassociator that stores a prescribed set of vector pairs. We will refer to these pairs as pattern pairs. A BAM network is very similar to a Hopfield network but has two layers of neurons in which layer $F_X$ has $n$ neurons and layer $F_Y$ has $p$ neurons. The recall process of the BAM is an iterative one starting with a stimulus pair in $F_X$ and $F_Y$. After a number of iterations, the patterns in $F_X$ and $F_Y$ converge to a fixed point which is desired to be one of the pattern pairs.

The BAM has three important features [3]. First, it performs both heteroassociative and autoassociative data recalls: the final state in $F_X$ represents the autoassociative recall, while the final state in $F_Y$ represents the heteroassociative recall. Second, the initial input can be presented in any one of the two layers. Last, the BAM is stable during recall.

To encode the pattern pairs, Kosko used the outer-product rule [3]. However, with the outer-product rule the memory capacity is very small if the pattern pairs are not orthogonal. Several modifications have been proposed to improve the memory capacity. These modifications fall into two categories:

1) modifying the encoding methods [4]–[6] and 2) introducing second-order connections to form the second-order bidirectional associative memory (SOBAM) [7]–[9]. The memory capacity of the SOBAM has been empirically studied [7] but the theoretical memory capacity has not yet been derived. The SOBAM has also been proven to be stable during recall [7].

This paper describes the stability and statistical properties of the SOBAM. Contrary to Simpson’s works [7], we demonstrate that the stability of the SOBAM is not guaranteed during recall. We also point out a mistake in [7]. This mistake has led to the wrong conclusion that the stability of the SOBAM is guaranteed. Hence, we cannot use the energy approach to estimate the statistical properties of the SOBAM, especially the memory capacity and the attraction basin. In this paper, we are interested in knowing whether each pattern pair can attract all the initial inputs within a certain distance from it. If so, we can obtain the attraction basin. Another important performance index is memory capacity, i.e., the maximum number of pattern pairs that can be stored in the SOBAM as attractors. Also of interest is the number of errors in the retrieved pairs. The question now is: given any $\rho_0$ errors in the initial input (an arbitrary error pattern with $\rho_0$ errors in the initial input), how does the number of errors vary during recall? To answer this question, we develop the statistical dynamics of the SOBAM. From this dynamics, the number of errors in the retrieved pairs, the attraction basin, and the memory capacity can be estimated.

Section II reviews the SOBAM and discusses its stability. The statistical dynamics of the SOBAM is developed in Section III, using the theory of large deviation [10]. Section IV discusses the way to estimate the memory capacity, the attraction basin, and the number of errors in the retrieved items. Numerical examples are given in Section V. Section VI shows how the results can be generalized to higher order bidirectional associative memories (HOBAM’s).

II. SOBAM AND STABILITY

There are $m$ pattern pairs $\{(X_1, Y_1), \cdots, (X_m, Y_m)\}$, where $X_h = (x_{1h}, \cdots, x_{nh})^T$ and $Y_h = (y_{1h}, \cdots, y_{mh})^T$. The components of $X_h$ and $Y_h$ are bipolar (+1 or -1). The SOBAM encodes the pattern pairs in two matrices. The first matrix, $U$, is a $n \times n \times p$ lattice that holds the second-order connections from $F_X$ to $F_Y$. The second matrix, $V$, is a $p \times p \times n$ lattice that holds the second-order connections from $F_Y$ to $F_X$. The matrix $U = [u_{k,j}]$ is given by

$$u_{k,j} = \sum_{h=1}^{m} y_{k,h} x_{j,h} x_{i,h} \quad \forall j,i = 1, \cdots, n, \ \text{and} \ \ k = 1, \cdots, p.$$
The matrix $V = [v_{jk}]$ is given by

$$v_{jk} = \sum_{l=1}^{m} x_{jl} y_{lk} y_{kh} \quad \forall k, l = 1, \ldots, p, \text{ and } j = 1, \ldots, n.$$  

The $F$ state at time $t$ is denoted as $X(t) = (x_1(t), \ldots, x_n(t))^T$. The $F$ state at time $t$ is denoted as $Y(t) = (y_1(t), \ldots, y_n(t))^T$.

The recall process is

$$y_{k}^{(t+1)} = \text{sgn} \left( \sum_{j=1}^{n} \sum_{i=1}^{n} u_{ijk} x_{j}^{(t)} x_{i}^{(t)} \right)$$  

for $k = 1, \ldots, p$, where

$$\text{sgn}(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \\ \text{state unchanged} & x = 0 \end{cases}$$

Similarly

$$x_{j}^{(t+1)} = \text{sgn} \left( \sum_{k=1}^{p} \sum_{l=1}^{p} u_{jkl} y_{k}^{(t+1)} y_{l}^{(t+1)} \right)$$

for $j = 1, \ldots, n$. Equations (3) and (4) imply that the initial input $X^{(0)}$ recalls $Y^{(4)}$; $Y^{(1)}$ recalls $X^{(1)}$ and so on.

The SOBAM is a finite-state autonomous system whose state converges to either a stable state or a limit cycle. Unlike the original BAM, the stability of the SOBAM is not guaranteed during recall. To illustrate this, we use a SOBAM network to store the following pattern pairs:

- $X_1 = (-1, -1, -1, -1, -1)^T$
- $Y_1 = (-1, -1, -1, -1, -1)^T$
- $X_2 = (1, -1, -1, -1, -1)^T$
- $Y_2 = (1, -1, -1, -1, -1)^T$
- $X_3 = (-1, -1, -1, -1, -1)^T$
- $Y_3 = (-1, -1, -1, -1, -1)^T$
- $X_4 = (-1, 1, -1, -1, -1)^T$
- $Y_4 = (1, 1, -1, -1, -1)^T$
- $X_5 = (1, 1, 1, 1, 1)^T$
- $Y_5 = (1, 1, 1, 1, 1)^T$.

With the initial state $X^{(0)} = (-1, -1, -1, -1, -1)^T$ and $Y^{(0)} = (1, 1, -1, -1, -1)^T$, the following states can be obtained, shown in the equation at the bottom of the page.

Clearly, the network converges to a limit cycle. Thus, the stability of the SOBAM is not guaranteed.

Simpson [7] used an energy function to explain the stability of the SOBAM. The energy function is expressed as

$$E_2 = (E_{2Y} + E_{2X})$$

$$= -\sum_{h=1}^{m} (X_h^T Y)^2 (X_h^T X) - \sum_{h=1}^{m} (Y_h^T Y)^2 (Y_h^T Y)$$

where $(X, Y)$ are the current states, $E_{2Y} = -\sum_{h=1}^{m} (X_h^T Y)^2 (X_h^T Y)$ represents the $F_Y$ energy, and $E_{2X} = -\sum_{h=1}^{m} (Y_h^T Y)^2 (Y_h^T Y)$ represents the $F_X$ energy. According to the recall process, either $F_X$ or $F_Y$ is updated first. If $F_X$ is updated first, the change in energy is

$$\Delta E_2 = \Delta E_{2X} = -\sum_{h=1}^{m} (Y_h^T Y)^2 (X_h^T \Delta X)$$

where $\Delta X = X^{\text{new}} - X$ and $X^{\text{new}}$ is the new state in $F_X$. Conversely, if $F_Y$ is updated first, the change in energy is

$$\Delta E_2 = \Delta E_{2Y} = -\sum_{h=1}^{m} (X_h^T X)^2 (Y_h^T \Delta Y)$$

where $\Delta Y = Y^{\text{new}} - Y$ and $Y^{\text{new}}$ is the new state in $F_Y$.

Simpson showed that the values of $\Delta E_{2X}$ and $\Delta E_{2Y}$ are either negative or zero. He then claimed that the SOBAM is always stable [7]. However, from our previous counter example, it can be seen that the stability is not guaranteed. This discrepancy is due to the omission of some terms on the right-hand side of (6) and (7). Actually, if $F_X$ is updated first, the total change in energy is

$$\Delta E_2 = -\sum_{h=1}^{m} (Y_h^T Y)^2 (X_h^T \Delta X) - \sum_{h=1}^{m} (X_h^T X)^2 (Y_h^T Y)$$

$$- \sum_{h=1}^{m} 2(X_h^T \Delta X)(X_h^T \Delta X)(Y_h^T Y).$$

---

\[
\begin{align*}
Y^{(1)} &= \text{sgn}(-8, 0, 8, -8, -8)^T &= (-1, 1, -1, -1, -1)^T \\
X^{(1)} &= \text{sgn}(-8, 32, -40, -8, 40)^T &= (-1, -1, -1, -1, -1)^T \\
Y^{(2)} &= \text{sgn}(-12, -52, 12, -84, 20)^T &= (-1, -1, -1, -1, -1)^T \\
X^{(2)} &= \text{sgn}(8, 48, -56, 8, -56, -24)^T &= (1, 1, -1, 1, 1, -1)^T \\
Y^{(3)} &= \text{sgn}(-12, -52, 12, -84, 20)^T &= (-1, -1, -1, -1, -1)^T \\
X^{(3)} &= \text{sgn}(8, 48, -56, 8, -56, -24)^T &= (1, 1, -1, 1, 1, -1)^T \\
Y^{(4)} &= \text{sgn}(-12, 12, -12, 20, -12, -52)^T &= (-1, -1, -1, -1, -1, -1)^T \\
X^{(4)} &= \text{sgn}(8, 48, -56, 8, 56, -24)^T &= (1, 1, -1, 1, 1, -1)^T \\
Y^{(5)} &= \text{sgn}(-12, 12, -12, 20, -12, -52)^T &= (-1, -1, -1, -1, -1, -1)^T \\
X^{(5)} &= \text{sgn}(8, 48, -56, 8, -56, -24)^T &= (1, 1, -1, 1, 1, -1)^T \\
Y^{(6)} &= \text{sgn}(-12, -52, 12, -84, 20, -52)^T &= (-1, 1, -1, -1, -1)^T \\
X^{(6)} &= \text{sgn}(8, 48, -56, 8, 56, -24)^T &= (1, 1, -1, 1, 1, -1)^T \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{align*}
\]
In this equation, the first term on the right-hand side is the change in \( E_{2X} \) due to a change of the \( F_X \) state. The other two terms, which represent the change in \( E_{2Y} \) due to a change of the \( F_X \) state, are either negative or positive. Hence, we cannot draw any conclusion regarding the stability based on the energy function proposed by Simpson [7].

The above discussion is valid for layer-synchronous recall process in which all neurons in a layer are updated simultaneously. Since layer-synchronous recall process is a special case of asynchronous recall processes whereby the neurons in a layer are updated sequentially, the stability of the SOBAM is not guaranteed under both layer-synchronous and asynchronous recall processes.

### III. Statistical Dynamics

#### A. Notations and Outline

This section outlines how the statistical dynamics is derived. We first define some terminologies and state the assumptions used in the rest of the paper.

- \( p = r\eta \), where \( r \) is a positive constant.
- \( m = \alpha n^2 \), where \( \alpha \) is a positive constant.
- The dimensions, \( n \) and \( p \), are large. This assumption is often used [10]–[15].
- For analytical purposes, we assume that each component of the pattern pairs is a ±1 equiprobable independent random variable. Though this assumption is not always being satisfied in most real-life data, it is difficult to analyze associative memories without making such an assumption. In fact, this assumption has been widely used [10]–[15].
- The Hamming distance between two bipolar vectors, \( X \) and \( X' \), is denoted as \( d(X, X') \).
- **Attraction Basin:** It is required that each pattern pair is stored as a stable state (or at least there is a stable state at a small Hamming distance). Otherwise, the pattern pairs cannot be recalled. Besides, we expect a SOBAM network to have the following error correction property. If the network is started at a state \( X^{(0)} \) where \( d(X_h, X^{(0)}) \leq \rho_x^{(0)}n \), the \( F_X \) state will reach a stable state within a distance of \( \rho_x^{(0)}n \) from the stored pattern \( X_h \) after a sequence of state transitions where \( \rho_x^{(0)} > \rho_x^{(f)} \) (the \( F_Y \) state should also reach a stable state within a distance of \( \rho_y^{(f)} \) from the stored pattern \( Y_h \) where \( \rho_x^{(0)} > \rho_y^{(f)} \)). We are interested in knowing whether each pattern pair is able to attract all the initial inputs \( X^{(0)} \) within a distance of \( \rho_x^{(0)}n \) for some positive constants \( \rho(0) \). The maximum value of such \( \rho(0) \) denotes the attraction basin of each pattern pair. Also of interest is the number of errors in the retrieved items. This number measures the quality of the retrieved items. Since we are considering “all possible initial inputs within a certain distance,” the above definition of the attraction basin is for worst case errors. In the rest of the paper, the term “attraction basin” refers to the attraction basin for worst case errors. Instead of estimating the attraction basin directly, we will estimate the number of errors after each state transition.

- Given that \( p = r\eta \) and \( m = \alpha n^2 \), \( P_{X}^{(t)} \) is the probability that for each pattern pair \((X_h, Y_h)\) and for any error pattern with \( \rho_x^{(t)}n \) errors in \( F_X \) in the present state \( d(X_h, X^{(t)}) = \rho_x^{(t)}n \), the number of errors in \( F_Y \) in the next state is less than \( \rho_y p \) \( d(Y_h, Y^{(t+1)}) < \rho_y p \). It should be noticed that the phrase “for any error pattern with \( \rho_x^{(t)}n \) errors in \( F_X \)” in the definition of \( P_{X}^{(t)} \) reflects the concept of worst case errors.
- Given that \( p = r\eta \) and \( m = \alpha n^2 \), \( P_{Y}^{(t)} \) is the probability that for each pattern pair \((X_h, Y_h)\) and for any error pattern with \( \rho_y^{(t+1)}p \) errors in \( F_Y \) in the present state \( d(Y_h, Y^{(t+1)}) = \rho_y^{(t+1)}p \), the number of errors in \( F_X \) in the next state is less than \( \rho_x n \) \( d(X_h, X^{(t+1)}) < \rho_x n \).

#### III. Statistical Dynamics

- The error rate in \( F_X \) is \( \frac{\text{The number of errors in } F_X}{n} \).
- The error rate in \( F_Y \) is \( \frac{\text{The number of errors in } F_Y}{p} \).
- To estimate the value of \( P_{Y}^{(t)} \), we first introduce the event \( E_{A_{h,g}} \). It is the event that \( d(Y^{(t+1)}, Y_h) < \rho_y p \) for a given pattern pair \((X_h, Y_h)\) and for a given present state \( X^{(t)} \in S_{h,t} \), where

\[
S_{h,t} = \left\{ X \in \{+1, -1\}^n \text{ such that } d(X, X_h) = \rho_x^{(t)}n \right\}.
\]

The index \( g \) refers to a particular error pattern. For a given \( \rho_x^{(t)} \), the number of error patterns is \( \binom{n}{\rho_x^{(t)}n} \). Thus, the range of \( g \) is from one to \( \binom{n}{\rho_x^{(t)}n} \). Also, \( E_{A_{h,g}} \) is the complement event of \( E_{A_{h,g}} \). It is the event that \( d(Y^{(t+1)}, Y_h) > \rho_y p \) for a given pattern pair \((X_h, Y_h)\) and for a given present state \( X^{(t)} \in S_{h,t} \).
- In the above, each event \( E_{A_{h,g}} \) only refers to an error pattern and a pattern pair. To consider each pattern pair and all the possible error patterns, we need to introduce the event \( E_A \) which is the intersection of all possible \( E_{A_{h,g}} \)’s

\[
E_A = \bigcap_{h,g} E_{A_{h,g}}.
\]

It is the event that

\[
d(Y^{(t+1)}, Y_h) < \rho_y p
\]

for each pattern pair \((X_h, Y_h)\) and for any \( X^{(t)} \in S_{h,t} \).
- Also, \( E_A \) is defined as the complement event of \( E_A \)

\[
\overline{E_A} = \bigcup_{h,g} \overline{E_{A_{h,g}}}.
\]
From the definitions of $P_X^{**}$ and $P_Y^{**}$

$$
P_X^{**} = \text{Prob}(E^A)$$

$$= 1 - \text{Prob}(\bigcup_{h,g} E_{A_{h,g}})$$

$$\geq 1 - \sum_{h=1}^{m} \sum_{g=1}^{c_{h+1}} \text{Prob}(E_{A_{g,h}}).$$

(10)

In Part B, we will first estimate the values of $P_X^{**}$ and $P_Y^{**}$ (Lemmas 4 and 5). From the two lemmas, an upper bound on the error rate in the next state is obtained (Corollaries 2 and 4). Based on Corollary 2, given that $\rho_x(t)$ is the upper bound on the error rate in $F_X$ at time $t$, we can derive an upper bound $\rho_y(t+1)$ on the error rate in $F_Y$ at time $(t+1)$. Similarly, from Corollary 4, we can estimate $\rho_y(t+1)$ from $\rho_y(t+1)$. As a result, two sequences $\{\rho_x(t)\}$ and $\{\rho_y(t)\}$ are constructed to represent the statistical dynamics of the SOBAM. In Section IV, we will discuss how to use the features of these two sequences to estimate the memory capacity, the number of errors in the retrieved pairs, and the attraction basin.

B. Construction of the Dynamics

To estimate the values of $P_X^{**}$ and $P_Y^{**}$, we make use of Stirling’s formula and the theory of large deviation [10]. Here, we restate them as the following two lemmas.

**Lemma 1—Stirling’s Asymptotic Formula for Factorial:** Let $n$ be a large integer and $\delta \in (0,0,5)$. Then

$$\binom{n}{x_n} \sim \exp \left\{ n\tilde{\mathcal{N}}(\delta) \right\}$$

where

$$\tilde{\mathcal{N}}(\delta) = -\delta \log \delta - (1 - \delta) \log(1 - \delta).$$

\(\Box\)

**Lemma 2—Newman’s Lemma:** Suppose $\chi_{1,N}, \chi_{2,N}, \cdots$ are, for each $N$, independent, identically distributed, and symmetric random variables satisfying

1)\n
$$\lim_{N \to \infty} \text{Var}(\chi_{1,N}) = \sigma^2 \in (0, \infty).$$

(11)

2) For some real $L > 2$ and $t_0 > 0$,

$$\limsup_{N \to \infty} \left\{ E\left[ \exp(t_0 \chi_{1,N} P^{1/L}) \right] \right\} < \infty$$

(12)

where $E[X]$ is the expectation operator.

For any $\gamma \in (0, \infty)$ and

$$\mathcal{R} = \frac{\gamma^2}{2\sigma^2}$$

(13)

a sufficient condition for

$$\text{Prob}\left( M^{-1} \sum_{s=1}^{M} \chi_{s,N} \geq \gamma M^{-1/2} \right) \leq \exp\left(-\mathcal{R}M^{1/2}t_0\right)$$

(14)
as $M, N \to \infty$, is

$$\gamma^2 t_0^{-2} < 2^{L-2}(\sigma^2 t_0)^L.\quad (15)$$

\(\Box\)

From Lemmas 1 and 2, we first estimate a bound on $\text{Prob}(E_{A_{h,g}})$. Let $\mathcal{R}$ be such that

$$\left( \frac{\sqrt{\rho_y(t) (1 - 2\rho_x(t))}^2}{\gamma \sqrt{3\mathcal{R}^{3/4}}} \right) < \frac{3}{2}$$

for $g = 1, \cdots, m$ and $h = 1, \cdots, m$.

\(\Box\)

**Proof of Lemma 3:** Without loss of generality, we assume that all the components of the pattern pair $(X_h, Y_h)$ are positive: $X_h = (1, \cdots, 1)^T$ and $Y_h = (1, \cdots, 1)^T$. Let $J$ be the set of indexes at which $X(t)$ and $X_h$ differ. For a given $X(t) \in S_{h,t}$, there is only one $J$ and $|J| = \rho_x(t)$. Let $K$ be a set of indexes of $Y(t)$ where $|K| = \rho_y$. Note that there are $\left( \binom{n}{\rho_y} \right)$ such sets. Event $E_{A_{h,h}}$ implies that there is at least one $K$ such that

$$\sum_{k \in K} \sum_{j=1}^{n} u_{k,j} x_{i}^{(t)} x_{j}^{(t)} < 0.$$

Hence

$$\text{Prob}(E_{A_{h,h}}) \leq \text{Prob}(\text{there is at least one } K, \text{ where } |K| = \rho_y \text{, such that })$$

$$\sum_{k \in K} \sum_{j=1}^{n} u_{k,j} x_{i}^{(t)} x_{j}^{(t)} < 0$$

\(\leq \left( \binom{n}{\rho_y} \right) \text{Prob}\left( \sum_{k \in K} \sum_{j=1}^{n} u_{k,j} x_{i}^{(t)} x_{j}^{(t)} < 0 \text{ for a given } K \right).\)  

(16)

Let

$$P' = \text{Prob}\left( \sum_{k \in K} \sum_{j=1}^{n} u_{k,j} x_{i}^{(t)} x_{j}^{(t)} < 0 \text{ for a given } K \right).$$

(17)

From (1), we have

$$P' = \text{Prob}\left( \rho_y (1 - 2\rho_x(t))^2 \leq \rho_y \right) \sum_{h \neq h} \sum_{k \in K} \sum_{j \in J} \left( \sum_{j' \in J} x_{i}^{(t)} - \sum_{j' \in J} x_{j'}^{(t)} \right)^2 < 0.\)  

(18)
One can easily find that
\[
E \left[ \sum_{k \in K} y_{kh} \left( \sum_{j \in J} x_{jh} - \sum_{j \in J} x_{jh} \right)^2 \right] = 0
\]
and
\[
E \left[ \left( \sum_{k \in K} y_{kh} \left( \sum_{j \in J} x_{jh} - \sum_{j \in J} x_{jh} \right)^2 \right)^2 \right] = \rho y (3n^2 - 2n).
\]
Hence
\[
P'' = \text{Prob} \left( \frac{1}{m-1} \sum_{h \neq h'} \chi_{h'} > \gamma (m-1)^{-1/4} \right)
\]
where
\[
\gamma = \frac{\sqrt{\rho y} (1 - 2 \rho x_{(2)})^2 n^{5/2}}{\sqrt{3n^2 - 2n}(m-1)^{3/4}}
\]
and
\[
\chi_{h'} = \frac{\sum_{k \in K} y_{kh} \left( \sum_{j \in J} x_{jh} - \sum_{j \in J} x_{jh} \right)^2}{\sqrt{3n^2 - 2n}(m-1)^{3/4}}.
\]
As \( n \) and \( p \to \infty \)
\[
\frac{m-1}{n} \to \alpha \quad \text{and} \quad \gamma \to \frac{\sqrt{\rho y} (1 - 2 \rho x_{(2)})^2}{\sqrt{3n^2 - 2n}(m-1)^{3/4}}.
\]
By substituting Lemma 2 (put \( L = 3 \)) into (19), we obtain
\[
P'' \leq \exp \left\{ -\frac{\rho y n (1 - 2 \rho x_{(2)})^4}{6\alpha} \right\}
\]
provided that
\[
\left( \frac{\sqrt{\rho y} (1 - 2 \rho x_{(2)})^2}{\sqrt{3n^2 - 2n}} \right)^4 < \frac{3}{2^4}
\]
for some real \( t_0 > 0 \). From the Appendix, \( t_0 \) must be less than \( 2^{3/3} \sqrt{3} / 4 \). To obtain a valid value of \( t_0 \), we have
\[
\left( \frac{\sqrt{\rho y} (1 - 2 \rho x_{(2)})^2}{\sqrt{3n^2 - 2n}} \right)^4 < \frac{3}{2^4}
\]
The procedures for checking whether \( \chi_{h'} \) satisfies the two conditions (11) and (12) are provided in the Appendix. By substituting (22) into (16), we obtain
\[
\text{Prob}(E_{A_{gh}}) \leq \frac{p_{x,p}}{p} \exp \left\{ -\frac{\rho y n (1 - 2 \rho x_{(2)})^4}{6\alpha} \right\}.
\]
From Lemma 1, we have
\[
\text{Prob}(E_{A_{gh}}) \leq \exp \left\{ \gamma \pi_{\rho y} - \frac{\rho y n (1 - 2 \rho x_{(2)})^4}{6\alpha} \right\}.
\]
Recall that
\[
P_{y}^{(4)} \geq 1 - \sum_{h \in H} \sum_{g \in G} \text{Prob}(E_{A_{gh}}).
\]
From Lemma 3
\[
P_{y}^{(4)} \geq 1 - m \left( \frac{n}{2} \right) \exp \left\{ \gamma \pi_{\rho y} - \frac{\rho y n (1 - 2 \rho x_{(2)})^4}{6\alpha} \right\}.
\]
With Lemma 1, we can immediately obtain the bound on \( P_{y}^{(4)} \) as Lemma 4 below.

Lemma 4:
\[
P_{y}^{(4)} \geq 1 - \exp \left\{ \gamma \pi_{\rho y} + \log cn^2 + m \gamma \pi_{\rho y} - \frac{\rho y n (1 - 2 \rho x_{(2)})^4}{6\alpha} \right\}
\]
provided that
\[
\left( \frac{\sqrt{\rho y} (1 - 2 \rho x_{(2)})^2}{\sqrt{3n^2 - 2n}} \right)^4 < \frac{3}{2^4}
\]
\( \Box \)

Let \( \rho_{y}^{*} \) be the minimum value of \( \rho_{y} \) such that the right-hand side of (26) tends to one (as \( n \to \infty \)). From Lemma 4, \( \rho_{y}^{*} \) is the minimum value of \( \rho_{y} \) such that
\[
\pi_{\rho_{y}^{*}} + \gamma \pi_{\rho_{y}} - \frac{\rho y n (1 - 2 \rho x_{(2)})^4}{6\alpha} < 0.
\]
Let \( \rho_{y}^{*} \) be the intersection of the line
\[
L_{1} : y = \frac{\rho y n (1 - 2 \rho x_{(2)})^4}{6\alpha} - \frac{\pi_{\rho y}}{\alpha}
\]
and the curve
\[
C_{1} : y = \pi_{\rho y}.
\]
Then
\[
\rho_{y}^{*} = \rho_{y}^{*} + \varepsilon
\]
where \( \varepsilon \) is an arbitrary small positive constant. Apparently, for a given \( \rho_{x}^{(2)} \in [0, 0.5) \), \( \rho_{y}^{*} \) can be solved numerically (see Fig. 1). Hence, for each pattern pair \( (x_{h}, y_{h}) \) and for any error pattern with \( \rho_{x}^{(2)} \) errors in \( X^{(2)} \) and \( \rho_{x}^{(2)} \) errors in \( Y^{(2)} \), the probability that the number of errors in \( Y^{(t+1)} \) is less than \( \rho_{y}^{*} \) tends to one (as \( n \to \infty \)). As \( \varepsilon \) can be any small positive constant, one can restate the above statement as: the probability that the number of errors in \( Y^{(t+1)} \) is less than or equal to \( \rho_{y}^{*} \) tends to one (as \( n \to \infty \)). Furthermore, according to the feature of \( L_{1} \) and \( C_{1} \), the following corollary can be obtained.
Fig. 1. Graphical implication of solving $\rho'_y$.

Corollary 1: If $\rho_{21}^{(t)} < \rho_{22}^{(t)} < 0.5$, then $\rho_{y1} < \rho_{y2}$.

Proof of Corollary 1: From (29) and (30) (see Fig. 1), if the value of $\rho_{22}^{(t)}$ is reduced, the line $L_2$ shifts up and its slope increases. Hence, the intersection shifts toward left and a smaller $\rho_y$ is obtained.

The above corollary implies that a smaller $\rho_y$ is obtained if the value of $\rho_{22}^{(t)}$ is reduced. Thus, given $\rho_{22}^{(t)} \in (0, 0.5)$, for each pattern pair $(X_i, Y_i)$ and for any $X^{(t)}$ such that $d(X_i, X^{(t)}) \leq \rho_{22}^{(t)} n$ (the error rate in $F_X$ in the present state is less than or equal to $\rho_{22}^{(t)}$), the probability that the error rate in $F_Y$ in the next state is less than or equal to $\rho_y$ tends to one (as $n \to \infty$). We capture the above statement as Corollary 2.

Corollary 2: Given that $\rho_{y}^{(t)}$ is the intersection of $L_1$ and $C_1$ (see (29) and (30)), for each pattern pair $(X_i, Y_i)$ and for any $X^{(t)}$ such that $d(X_i, X^{(t)}) \leq \rho_{22}^{(t)} n$, the probability that $d(Y_i, Y^{(t+1)}) \leq \rho_y^{(t+1)} n$ tends to one (as $n \to \infty$).

It follows from Corollary 2 that, if the error rate in $F_X$ in the present state is less than or equal to $\rho_{22}^{(t)}$, the error rate in $F_Y$ in the next state is less than or equal to $\rho_y^{(t+1)}$. Thus, $\rho_y^{(t+1)}$ defines an upper bound on the error rate in $F_Y$ in the next state. We denote this upper bound as $\rho_y^{(t+1)}$.

Similarly, we can easily obtain the lower bound on $P_X^{**}$ as Lemma 5.

Lemma 5:

$$P_X^{**} \geq 1 - \exp\left\{\rho_y^{(t+1)} + p \rho (1 - 2 \rho_y^{(t+1)}) \right\}$$

provided that

$$\left(\frac{\sqrt{\rho \sigma (1 - 2 \rho_y^{(t+1)})}}{\sqrt{3}}\right)^4 \leq \frac{3}{2}.$$  (32)

From Lemma 5, we can estimate the error rate $\rho'_y$ in $F_X$ in the next state (given the error rate in $F_Y$ in the present state) by considering the intersection of the line

$$L_2 : y = \rho_x^{(t)} (1 - 2 \rho_y^{(t+1)}) - \tau \rho (\rho_y^{(t+1)})$$  (33)

and the curve

$$C_2 : y = \pi (\rho_x^{(t)}).$$  (34)

Hence, Corollaries 3 and 4 are obtained.

Corollary 3: If $\rho_{y1}^{(t+1)} < \rho_{y2}^{(t+1)} < 0.5$, then $\rho_{x2}^{(t+1)} < \rho_{x1}^{(t+1)}$.

Corollary 4: Given that $\rho_x^{(t+1)}$ is the intersection of $L_2$ and $C_2$ (see (33) and (34)), for each pattern pair $(X_i, Y_i)$ and for any $Y^{(t+1)}$ such that $d(Y_i, Y^{(t+1)}) \leq \rho_y^{(t+1)} n$, the probability that $d(X_i, X^{(t+1)}) \leq \rho_{y1}^{(t+1)}$ tends to one (as $n \to \infty$).

IV. ESTIMATION OF STATISTICAL PROPERTIES

A. Memory Capacity

Clearly, if the sequences $(\rho_x^{(t)}, \rho_y^{(t)})$, with some $\rho_{y1}^{(t)}$, converge to two small numbers individually, each pattern pair can attract all initial inputs within a certain distance. In other words, each pattern pair or its noisy versions can be stored as a stable state. Therefore, we can use the following method to estimate the memory capacity.

For a given $n$, we can estimate the memory capacity by using the sequences $(\rho_x^{(t)}, \rho_y^{(t)})$ with some $\rho_{y1}^{(t)}$, and $\rho_y^{(t+1)}$, respectively (where $\rho_x^{(t)} > \rho_y^{(t)}$). Thus, $\rho_y^{(t+1)}$ can be considered as a lower bound on the memory capacity.
Another error correction index is the attraction basin for random errors. In this case, we need to find out if a stored item can attract “an” initial input within a certain distance. The terms “an” and “every” mark the difference between worst case errors and random errors. It should be noticed that using simulations to study the attraction basin for worst case errors is impractical. The reason is that the number of error patterns is very large (for example, \(\binom{1000}{3} \approx 5 \times 10^{7}\)). Simulations can only reflect the properties of models in the presence of random errors.

In the case of random errors, if the number of the pattern pairs is less than

\[
\min \left( \frac{(1-2\rho)^2n^2}{18\log n}, \frac{p^2}{18\log p} \right)
\]

an initial input (in \(F_X\)) within distance \(\rho_n\) from \(X_h\) will be attracted to the desired pattern pair in two recall steps with high probability. The proof of the above behavior is based on the theory of large deviation (see \[19, Lemma A.7\]). As this paper is mainly concerned with worst case errors, the behavior of the models in the presence of random errors will not be discussed further.

D. Approximations of \(\rho^f_X\) and \(\rho^f_Y\)

It is not difficult to see the following relationship between \(\rho^f_X\) and \(\rho^f_Y\).

Corollary 5: If the sequences \((\{\rho^f_x(t)\}, \{\rho^f_y(t)\})\) converge to small \(\rho^f_x(<< 1)\) and small \(\rho^f_y(<< 1)\), respectively, then

\[
\rho^f_y \approx \tau \rho^f_x \cdot
\]

Proof of Corollary 5: From (29), (30), (33), and (34), we have

\[
\hat{\eta}(\rho^f_y) = \frac{\rho^f_y(1-2\rho^f_y)^4}{6\alpha} - \frac{\hat{\eta}(\rho^f_x)}{r}.
\]

Thus

\[
\frac{\rho^f_y(1-2\rho^f_y)^4}{\rho^f_x(1-2\rho^f_x)^4} = r.
\]

As the values of \(\rho^f_X\) and \(\rho^f_Y\) are small

\[
\rho^f_y \approx \tau \rho^f_x.
\]

Similarly, we can easily obtain Corollary 6, which can be used for the direct estimation of \(\rho^f_X\) and \(\rho^f_Y\).

Corollary 6: If the values of \(\rho^f_X\) and \(\rho^f_Y\) are small (<< 1), we have

\[
\rho^f_X \approx \exp \left\{ -\frac{r^2}{(1+r^2)\alpha \bar{r}} + 1 - \frac{r^2}{1+r^2} \log r \right\} \quad \text{and} \quad \rho^f_Y \approx \exp \left\{ -\frac{r^2}{(1+r^2)\alpha \bar{r}} + 1 + \frac{1}{1+r^2} \log r \right\}.
\]
Proof of Corollary 6: From (28), we have

\[ \hat{h}(\rho^L) + r\hat{h}(\rho^L) \approx \frac{\rho^L(1 - 2\rho^L)^{1/4}}{6\alpha}. \]

Using the approximation “\(\log(1 - \rho) \approx -\rho\) for small positive \(\rho\),” the corollary follows.

Remark: Since we are concerned with the conditions under which the lower bounds on \(\alpha\) and \(\gamma\) tend to one (as \(n \to \infty\)), we only obtain the bounds on the three statistical properties: the memory capacity, the number of errors in the retrieved pairs and the attraction basin. The actual values of these three statistical properties considered in this section are better than the estimated bounds. It should also be noticed that during the construction of the sequences \((\{\rho^x\}, \{\rho^y\})\), we should check whether both conditions (27) and (32) are satisfied.

V. NUMERICAL RESULTS

Numerical Example a: Based on the theoretical work presented in the previous section, we estimate the lower bound on the memory capacity. The estimated results are summarized in Table I. The lower bound increases with \(r\) until \(r = 10\) where it starts to decrease. Also

\[ \alpha_r \approx \frac{\alpha_r}{\sqrt{r^2}}. \]

This symmetrical property means that inverting the ratio of the dimensions (interchange \(p\) and \(n\)) does not affect the overall estimated lower bound.

One should be aware that when the values of \(n\) and \(p\) are small, the bound may become meaningless. For example, if \(n = p = 10\) and \(r = 1\), the lower bound is about 1.28 (which is less than \(n = 10\)). However, for large \(n\) and \(p\), the result is different. For example, if \(n = 10^3\) and \(r = 1\), the lower bound is about \(1.28 \times 10^3\) (which is much greater than the dimension \(n = 10^3\)). For image processing problems [17], the dimensions are usually greater than \(10^3\).

Fig. 3. The lower bound on the attraction basin where \(r = 1, 2, 5, 10\).

Numerical Example b:

Fig. 3 summarizes the lower bounds on the attraction basin at \(r = 1, 2, 5, 10\). When the value of \(\alpha\) is small, the lower bound first increases as \(\gamma\) decreases. However, when the value of \(\alpha\) is too small, the lower bound starts to decrease as \(\alpha\) further decreases. This unnatural trend is due to the constraints (27) and (32), which limit the searching range of \((\rho^x, \rho^y)\). However, it is rational to accept the claim that for a smaller \(\alpha\), a larger attraction basin is obtained. We take the maximum point in the figure as the lower bound on the attraction basin for small values of \(\alpha\). Table II summarizes the above claim. From Fig. 3 and Table II, for a meaningful attraction basin, the dimension \(n\) should be larger than \(10^3\). The estimated attraction basin is quite small. This is not surprising because the estimated lower bounds refer to worst case error.

Figs. 4 and 5 show the behavior of \(\rho^L_x\) and \(\rho^L_y\). From the figures, the upper bound on the number of errors in the retrieved pairs (\(\rho^L_x\) or \(\rho^L_y\)) decreases exponentially as \(\alpha\) decreases (this feature matches Corollary 6). Also, the value of \(\rho^L_y\) is approximately equal to that of \(r\rho^L_x\) (as was indicated in Corollary 5). Since it is desired that the number of retrieved errors should be as small as possible, the estimated upper bounds are more attractive.

VI. HOBAM’S

Although we are mainly concerned with the properties of the SOBAM, we can apply a similar method to analyze HOBAM’s. The only required change in the assumptions is

\[ m = \alpha \eta^d \]

(39)
where $q$ is a positive integer. For the $q$-order BAM, the connections from $F_X$ to $F_Y$ are

$$u_{k_1, k_2, \ldots, k_q} = \sum_{l=1}^{m} y_{k_l} x_{i_1} x_{i_2} \cdots x_{i_q}$$

(40)

where $k_1, \ldots, k_q, i_1, \ldots, i_q$ are positive integers. The connections from $F_Y$ to $F_X$ are

$$v_{j_1, j_2, \ldots, j_q} = \sum_{l=1}^{m} x_{j_l} y_{j_1} y_{j_2} \cdots y_{j_q}$$

(41)

where $j_1, \ldots, j_q, l_1, \ldots, l_q, p_1, \ldots, p_q$ are positive integers. The corresponding recall equations are

$$y_{k_l}^{(t+1)} = \text{sgn}\left(\sum_{i_1, \ldots, i_q=1}^{n} u_{k_1, k_2, \ldots, k_q} x_{i_1}^{(t)} x_{i_2}^{(t)} \cdots x_{i_q}^{(t)}\right)$$

(42)

and

$$x_{j_l}^{(t+1)} = \text{sgn}\left(\sum_{l_1, \ldots, l_q=1}^{p} v_{j_1, j_2, \ldots, j_q} y_{l_1}^{(t)} y_{l_2}^{(t)} \cdots y_{l_q}^{(t)}\right).$$

(43)

We can obtain similar results for the $q$-order BAM based on the following lemma.

**Lemma 6:** Let $\xi_1, \xi_2, \ldots, \xi_n$ be $\pm 1$ equiprobable independent random variables and $S_n^q = \sum_{i=1}^{n} \xi_i$, where $n$ is a positive integer. As $n \to \infty$

$$E\left[(S_n^q)^{2q}\right] \to \frac{(2q)!}{2^{2q} q!}.$$

(44)

**Proof of Lemma 6:** As $n \to \infty$, the distribution of $S_n^q$ tends to be normal. Since the $2q$th moment of a normal random variable [18] is $1 \cdot 3 \cdots (2q - 1)$

$$E\left[(S_n^q)^{2q}\right] \to 1 \cdot 3 \cdots (2q - 1)$$

$$= \frac{(2q)!}{2^{2q} q!}.$$  

(44)

From Lemma 6, we can obtain the four corollaries for the $q$-order BAM.

**Corollary 7:** For every pattern pair $(X_h, Y_h)$ and for any $X^{(t)}$ such that $d(X_h, X^{(t)}) \leq \rho_n^{(t)}$, the probability that $d(Y_h, Y^{(t+1)}) \leq \rho_n^{(t+1)}$ tends to one (as $n \to \infty$), provided that

$$\frac{\sqrt{\kappa} \left(1 - \frac{\rho_n^{(t)}}{\rho_n^{(t+1)}}\right) \lambda_t}{\sqrt{\kappa} \left(\frac{\rho_n^{(t+1)}}{\rho_n^{(t)}}\right)} \frac{2q}{\lambda_t^2} < \frac{\lambda_t}{2}$$

and

$$\rho_n^{(t)} < \rho_n^{(t+1)}.$$
\[ \frac{t_k}{k!} E[|\chi| |\omega|] \leq C_6 e^{-((\sigma+1))\omega/2-1})k((2\omega/2\Lambda)^{1/2})k((\omega_k + 1)/2)k^{-1/2}. \]  

\[ (54) \]

where

\[ \lambda_q = \frac{(2q)!}{2\sqrt{q}!} \]

and \( \rho_y \) is the intersection of \( \Lambda_{2q} \) and \( C_{2q} \)

\[ \Lambda_{2q} : y = \frac{\rho_y(1 - 2\rho_y^{(1)})^{2q}}{2\lambda_q^{(q)} \gamma} - \frac{\pi(\rho_y^{(1)})}{r} \]

\[ C_{2q} : y = \hat{h}(\rho_y). \]

\[ \text{Corollary 8:} \] For every pattern pair \((X_k, Y_k)\) and for any \( Y^{(t+1)} \) such that \( d(Y_k, Y^{(t+1)}) \leq \rho_y^{(t+1)} \), the probability that \( d(X, Y^{(t+1)}) \) is the intersection of \( L_{2q} \) and \( C_{2q} \)

\[ \Lambda_{2q} : y = \frac{\rho_y^{2q}(1 - 2\rho_y^{(1)})^{2q}}{2\lambda_q^{(q)} \gamma} - \gamma(\rho_y^{(1)}) \]

\[ C_{2q} : y = \hat{h}(\rho_y). \]

\[ \text{Corollary 9:} \] If the sequences \( \{\rho_y^{(t)}\}, \{\rho_y^{(1)}\} \) converge to small \( \rho_y^{(t)}(1 < 1) \) and small \( \rho_y^{(1)}(1 < 1) \), respectively, then

\[ \rho_y^{(t)} \approx r^{\gamma - 1} \rho_y^{(1)}. \]

\[ (45) \]

\[ \text{Corollary 10:} \] If the values of \( \rho_y^{(t)} \) and \( \rho_y^{(1)} \) are small \((1 < 1)\), we have

\[ \rho_y^{(t)} \approx \exp \left\{ -\frac{r^{(q)}}{1 + r^{(q)}} 2\lambda_q^{(q)} \gamma \right\} + \frac{r^{(q)}}{1 + r^{(q)}} \log r^{(q) - 1} \]

\[ (46) \]

and

\[ \rho_y^{(1)} \approx \exp \left\{ -\frac{r^{(q)}}{1 + r^{(q)}} 2\lambda_q^{(q)} \gamma \right\} + \frac{1}{1 + r^{(q)}} \log r^{(q) - 1}. \]

\[ (47) \]

\[ \text{VII. Concluding Remarks} \]

In this paper, we have studied several properties of the SOBAM. An example has been given which shows that the SOBAM may not be stable during recall. We have also derived the statistical dynamics of the SOBAM. Based on this dynamics, we have estimated the memory capacity, the attraction basin and the number of errors in the retrieved pairs. Numerical examples have also been presented. Last, we have briefly discussed how our results can be extended to HOBAM’s. One significant advantage of the methodology presented is that we can analyze some associative memories whose stability is not guaranteed.

\[ \text{APPENDIX} \]

\[ \text{Verification of the Conditions of Newman’s Lemma} \]

Here, we show the conditions under which the random variable

\[ \chi = \frac{\left( \sum_{k=1}^n x_k j_k \right)^{\alpha}}{\sqrt{\rho_0 \beta \rho_{0}^{(n)}}} \]

satisfies (11) and (12), where \( y_k \)’s and \( x_j \)’s are \( \pm 1 \) equiprobable independent random variables. Clearly, \( \chi \) is symmetric and

\[ E[\chi] = 0. \]

Also, from Lemma 6

\[ \text{Var}(\chi) = E[\chi^2] = 1. \]

\[ (48) \]

\[ (49) \]

Hence, (11) is satisfied.

To check whether \( \chi \) satisfies (12), we use an existing result about the sum of \( \pm 1 \) equiprobable independent random variables [19].

\[ \text{Lemma 7:} \] Let \( \xi_1, \xi_2, \ldots, \xi_n \) be \( \pm 1 \) equiprobable independent random variables. For \( z > 0 \) and large \( n \)

\[ E[\left( \sum_{k=1}^n \xi_k j_k \right)^2] \leq 2z/2^{1/2} \Gamma(z + 1/2) \]

\[ (50) \]

where \( \Gamma(a) \) is the gamma function

\[ \Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx. \]

\[ (51) \]

\[ \text{The above lemma is part of [19, Lemma A.6]. From Lemma 7, we have} \]

\[ E[\chi^2] \leq 2z/2^{1/2} \Gamma(z + 1/2) \Gamma(z + 1/2) \lambda^{z/2}. \]

\[ (52) \]

Let \( z = \omega k \); where \( k \) is a positive integer and \( \omega > 0 \). Then

\[ \sqrt{k!} E[\chi] \leq C_o \frac{\sqrt{\rho_0 \beta \rho_{0}^{(n)}}}{k!} \Gamma(\frac{\omega + 1}{2}) \Gamma(\frac{\omega k + 1}{2}) \lambda^{-\omega k / 2} \]

\[ (53) \]

where \( C_o \) is a positive constant. For large \( k \)

\[ k! \Gamma(\frac{\omega k + 1}{2}) \approx \sqrt{2\pi} e^{-\omega k / 2} \Gamma(\frac{\omega k + 1}{2}) \Gamma(\frac{\omega k + 1}{2} - 1/2) \]

\[ (54) \]

and

\[ \Gamma(\frac{\omega k + 1}{2}) \approx \sqrt{2\pi} e^{-\omega k / 2} \Gamma(\frac{\omega k + 1}{2}) \Gamma(\frac{\omega k + 1}{2} - 1/2) \]

\[ (55) \]

Hence we have (54), shown at the top of the page.

For large \( k \), the \( k \)th term of the sum

\[ S = \sum_{k=0}^\infty \frac{t_k}{k!} E[\chi | \omega^k] \]
decreases exponentially provided that \( \omega = \frac{2}{q+1} \) and \( t_o < 2^{-\left(\frac{q}{(q+1)}\right)\frac{1}{2(q+1)}} \). As \( S \) converges to
\[
E\left[\exp\left\{t_o \mid \chi \right\}^{2/(q+1)}\right]
\]
it follows that
\[
\limsup_{n \to \infty} E\left[\exp\left\{t_o \mid \chi \right\}^{2/(q+1)}\right] < \infty.
\]
For the SOBAM, \( q = 2 \) and \( t_o < 2^{-2/3}3^{1/3} \).

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Edmund Man Kit Lai, photograph and biography not available at the time of publication.