Stability, Capacity, and Statistical Dynamics of Second-Order Bidirectional Associative Memory

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Abstract—**Bidirectional Associative Memory (BAM) is used for storage of bipolar library pairs. Second-order BAM is an enhanced version of BAM. The stability, capacity and statistical dynamics of second-order BAM are presented here. We first use an example to illustrate that the state of second-order BAM may converge to limited cycles. When error in the retrieved pairs is not allowed, a lower bound of memory capacity is derived. That is $O(\min(\frac{n^2}{\log n}, \frac{p^2}{\log p}))$, where n and p are the dimensions of the library pairs. Since the state of second-order BAM may converge to limited cycles, the conventional method cannot be used to estimate its memory capacity when small errors in the retrieval pairs are allowed. Hence, the statistical dynamics of second-order BAM is introduced: starting with an initial state close to the library pairs (there are some errors in the initial state), how the confidence interval of the number of errors changes during recalling. From the dynamics, the attraction basin, memory capacity, and final error in the retrieval pairs can be estimated. Also, some numerical results are given. Finally, extension of the results to higher-order BAM is discussed.**

I. Introduction

Associative memory is one of the major research issues in neural networks with a wide range of applications such as content addressable memory and pattern recognition [1], [2]. Bidirectional Associative Memory (BAM), which is a generalization of the Hopfield network [3], proposed by Kosko [4]. It is a heteroassociative memory that stores bipolar library pairs, $(\mathbf{X}_h, \mathbf{Y}_h)$, $h = 1, \cdots, m$, where $\mathbf{X}_h \in \{+1, -1\}^n$, $\mathbf{Y}_h \in \{+1, -1\}^p$, and m is the number of the library pairs. There are two layers in BAM. Layer F_X has n neurons to hold the vector $\mathbf{X}$ and layer F_Y has p neurons to hold the vector $\mathbf{Y}$.

The connection matrix $\mathbf{W}$, proposed by Kosko, is

$$\mathbf{W} = \sum_{h=1}^{m} \mathbf{Y}_h \mathbf{X}_h^T \tag{1}$$

where $\mathbf{X}_h = (x_{1h}, x_{2h}, \cdots, x_{nh})^T$ and $\mathbf{Y}_h = (y_{1h}, y_{2h}, \cdots, y_{ph})^T$. The retrieval process is an iterative process starting with a stimulus pair $(\mathbf{X}^{(0)}, \mathbf{Y}^{(0)})$ in F_X and F_Y. The vector $\mathbf{Y}^{(t+1)}$ in F_Y is generated by using $\mathbf{W}$ (the superscript (t) is the iteration index)

$$\mathbf{Y}^{(t+1)} = \mathrm{sgn}\left[\mathbf{W}\mathbf{X}^{(t)}\right] \tag{2}$$

where sgn is the sign operator

$$\mathrm{sgn}(x) = \begin{cases} +1, & x > 0 \\ -1, & x < 0 \\ \text{state unchanged}, & x = 0 \end{cases}.$$

Then $\mathbf{Y}^{(t+1)}$ is fed backward to generate the new state of F_X

$$\mathbf{X}^{(t+1)} = \mathrm{sgn}\left[\mathbf{W}^T \mathbf{Y}^{(t+1)}\right]. \tag{3}$$

Kosko [4] proved that the sequence $(\mathbf{X}^{(t)}, \mathbf{Y}^{(t)})$ converges to one of the fixed points $(\mathbf{X}_f, \mathbf{Y}_f)$ in a finite number of iterations. Usually, this fixed point is desired to be one of the library pairs. A fixed point $(\mathbf{X}_f, \mathbf{Y}_f)$ has the following properties

$$\mathbf{Y}_f = \mathrm{sgn}(\mathbf{W}\mathbf{X}_f) \quad \text{and} \quad \mathbf{X}_f = \mathrm{sgn}(\mathbf{W}^T \mathbf{Y}_f). \tag{4}$$

Obviously, a library pair can be retrieved only if it is a fixed point. Unfortunately, with Kosko's encoding method, the memory capacity of BAM is very small [5]–[7]. To improve the memory capacity, several encoding methods have been developed. These methods fall into two categories: either by they modify the connection matrix [5]–[7], or they introduce higher-order connections [8] (also called higher-order BAM). In [8], Simpson has empirically studied the memory capacity of second-order BAM. But the theoretical memory capacity and the statistical dynamics have not been given.

In the first portion of this paper, we will review second-order BAM with an example to demonstrate that the state of second-order BAM may converge to limited cycles. In the remainder of this paper, we theoretically estimate a lower bound of memory capacity of second-order BAM when no error in the retrieval pairs is tolerated. Under the assumption that $m = \alpha n^2$, we have developed the statistical dynamics of second-order BAM. With such statistical dynamics, we can estimate the memory capacity when small errors in the retrieval library pairs are allowed. Additionally, the attraction basin and final error in the retrieval pairs can be estimated. Finally, we will discuss how to generalize the above results to higher-order cases.

II. Second-Order BAM

Second-order BAM encodes library pairs into two separate matrices [8, 10]. The first matrix, $\mathbf{U}$, is a $n \times n \times p$ lattice that holds the second-order connections from F_X to F_Y. The second matrix, $\mathbf{V}$, is a $p \times p \times n$ lattice that holds the connections from F_Y to F_X. The matrix $\mathbf{U} = [u_{kji}]$ is constructed according to the correlation rule

$$u_{kji} = \sum_{h=1}^{m} y_{kh} x_{jh} x_{ih} \qquad \text{for } j = 1, \cdots, n,\ i = 1, \cdots, n, \text{ and } k = 1, \cdots, p. \tag{5}$$

Also, the matrix $\mathbf{V} = [v_{jkl}]$ is

$$v_{jkl} = \sum_{h=1}^{m} x_{jh} y_{kh} y_{lh} \qquad \text{for } l = 1, \cdots, p,\ k = 1, \cdots, p, \text{ and } j = 1, \cdots, n. \tag{6}$$

Note that $v_{jkl} = v_{jlk}$ and $u_{kji} = u_{kij}$. The recalling process of second-order BAM works in the same fashion as first-order BAM. That is

$$y_k^{(t+1)} = \mathrm{sgn}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} u_{kji} x_j^{(t)} x_i^{(t)}\right) \tag{7}$$

$$x_j^{(t+1)} = \mathrm{sgn}\left(\sum_{k=1}^{p}\sum_{l=1}^{p} v_{jkl} y_k^{(t+1)} y_l^{(t+1)}\right). \tag{8}$$

There is another definition of second-order BAM. The connections of this new model are

$$u_{kji} = \sum_{h=1}^{m} y_{kh} x_{jh} x_{ih} \qquad \text{for } k = 1, \cdots, p,\ j = 1, \cdots, n-1, \text{ and } i = j+1, \cdots, n, \tag{9}$$

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and

$$v_{jkl} = \sum_{h=1}^{m} x_{jh} y_{kh} y_{lh} \quad \text{for } j = 1, \cdots, n,\ k = 1, \cdots, p-1, \text{ and } l = k+1, \cdots, p. \tag{10}$$

The corresponding recalling rules are

$$y_k^{(t+1)} = \operatorname{sgn}\left(\sum_{j=1}^{n-1} \sum_{i>j}^{n} u_{kji} x_j^{(t)} x_i^{(t)} \right) \tag{11}$$

$$x_j^{(t+1)} = \operatorname{sgn}\left(\sum_{k=1}^{p-1} \sum_{l>k}^{p} v_{jkl} y_k^{(t+1)} y_l^{(t+1)} \right). \tag{12}$$

To distinguish the two different models, the first model is called total-order-connection second-order BAM. The other is called partial-order-connection second-order BAM. Although the two models are different, their statistical behaviors are similar. These will be shown in Sections IV and V.

III. Stability of Second-Order BAM

Since the number of states of second-order BAM is finite and the next state only depends on its present state, both models of second-order BAM belong to the class of finite-state autonomous systems. One can easily verify that the state of a finite-state autonomous system either converges to fixed points or limited cycles.

Unlike first-order BAM, the state of second-order BAM may converge to limited cycles. That means that the sequence $(\mathbf{X}^{(t)}, \mathbf{Y}^{(t)})$ may not converge to fixed points. Let us see the following example.

Consider the following library pairs

$$\begin{aligned}
\mathbf{X}_1 &= (-1, 1, -1, 1, 1, -1)^T \\
\mathbf{Y}_1 &= (-1, 1, 1, -1, 1, -1)^T \\
\mathbf{X}_2 &= (1, 1, -1, -1, 1, -1)^T \\
\mathbf{Y}_2 &= (-1, -1, 1, -1, -1, -1)^T \\
\mathbf{X}_3 &= (-1, -1, -1, -1, 1, -1)^T \\
\mathbf{Y}_3 &= (-1, -1, 1, -1, -1, 1)^T \\
\mathbf{X}_4 &= (-1, 1, -1, -1, 1, -1)^T \\
\mathbf{Y}_4 &= (1, -1, -1, -1, 1, -1)^T \\
\mathbf{X}_5 &= (1, 1, -1, 1, 1, 1)^T \\
\mathbf{Y}_5 &= (1, 1, -1, 1, -1, -1)^T
\end{aligned}$$

and the initial state is

$$\begin{aligned}
\mathbf{X}^{(0)} &= (-1, -1, 1, -1, 1, 1)^T \\
\mathbf{Y}^{(0)} &= (1, 1, -1, 1, 1, 1)^T.
\end{aligned}$$

When the total-order-connection second-order BAM is used, the following sequence can be obtained

$$\begin{aligned}
\mathbf{Y}^{(1)} &= \operatorname{sgn}(-8, 0, 8, -8, -8, -8)^T \\
&= (-1, 1, 1, -1, -1, -1)^T \\
\mathbf{X}^{(1)} &= \operatorname{sgn}(-8, 32, -40, -8, 40, -40)^T \\
&= (-1, 1, -1, -1, 1, -1)^T \\
\mathbf{Y}^{(2)} &= \operatorname{sgn}(-12, -52, 12, -84, 20, -52)^T \\
&= (-1, -1, 1, -1, 1, -1)^T \\
\mathbf{X}^{(2)} &= \operatorname{sgn}(8, 48, -56, 8, 56, -24)^T \\
&= (1, 1, -1, 1, 1, -1)^T \\
\mathbf{Y}^{(3)} &= \operatorname{sgn}(-12, 12, 12, -20, -12, -52)^T \\
&= (-1, 1, 1, -1, -1, -1)^T \\
\mathbf{X}^{(3)} &= \operatorname{sgn}(-8, 32, -40, -8, 40, -40)^T \\
&= (-1, 1, -1, -1, 1, -1)^T \\
\mathbf{Y}^{(4)} &= \operatorname{sgn}(-12, -52, 12, -84, 20, -52)^T \\
&= (-1, -1, 1, -1, 1, -1)^T \\
\mathbf{X}^{(4)} &= \operatorname{sgn}(8, 48, -56, 8, 56, -24)^T \\
&= (1, 1, -1, 1, 1, -1)^T \\
\mathbf{Y}^{(5)} &= \operatorname{sgn}(-12, 12, 12, -20, -12, -52)^T \\
&= (-1, 1, 1, -1, -1, -1)^T \\
\mathbf{X}^{(5)} &= \operatorname{sgn}(-8, 32, -40, -8, 40, -40)^T \\
&= (-1, 1, -1, -1, 1, -1)^T \\
\mathbf{Y}^{(6)} &= \operatorname{sgn}(-12, -52, 12, -84, 20, -52)^T \\
&= (-1, -1, 1, -1, 1, -1)^T \\
\mathbf{X}^{(6)} &= \operatorname{sgn}(8, 48, -56, 8, 56, -24)^T \\
&= (1, 1, -1, 1, 1, -1)^T \\
&\ \ \vdots
\end{aligned}$$

Clearly, the sequence $(\mathbf{X}^{(t)}, \mathbf{Y}^{(t)})$ converge to a limited cycle. In the partial-order-connection case, we get the following sequence

$$\begin{aligned}
\mathbf{Y}^{(1)} &= \operatorname{sgn}(-1, 3, 1, 5, -1, 5)^T \\
&= (-1, 1, 1, 1, -1, 1)^T \\
\mathbf{X}^{(1)} &= \operatorname{sgn}(-17, 7, -5, -17, 5, -11)^T \\
&= (-1, 1, -1, -1, 1, -1)^T \\
\mathbf{Y}^{(2)} &= \operatorname{sgn}(-3, -23, 3, -33, 13, 17)^T \\
&= (-1, -1, 1, -1, 1, 1)^T \\
\mathbf{X}^{(2)} &= \operatorname{sgn}(7, 15, -13, 7, 13, -3)^T \\
&= (1, 1, -1, 1, 1, -1)^T \\
\mathbf{Y}^{(3)} &= \operatorname{sgn}(-3, 9, 3, -1, -3, -17)^T \\
&= (-1, 1, 1, -1, -1, -1)^T \\
\mathbf{X}^{(3)} &= \operatorname{sgn}(-1, 7, -5, -1, 5, -11)^T \\
&= (-1, 1, -1, -1, 1, -1)^T \\
\mathbf{Y}^{(4)} &= \operatorname{sgn}(-3, -23, 3, -33, 13, 17)^T \\
&= (-1, -1, 1, -1, 1, 1)^T \\
\mathbf{X}^{(4)} &= \operatorname{sgn}(7, 15, -13, 7, 13, -3)^T \\
&= (1, 1, -1, 1, 1, -1)^T \\
\mathbf{Y}^{(5)} &= \operatorname{sgn}(-3, 9, 3, -1, -3, -17)^T \\
&= (-1, 1, 1, -1, -1, -1)^T \\
\mathbf{X}^{(5)} &= \operatorname{sgn}(-1, 7, -5, -1, 5, -11)^T \\
&= (-1, 1, -1, -1, 1, -1)^T \\
\mathbf{Y}^{(6)} &= \operatorname{sgn}(-3, -23, 3, -33, 13, 17)^T \\
&= (-1, -1, 1, -1, 1, 1)^T \\
\mathbf{X}^{(6)} &= \operatorname{sgn}(7, 15, -13, 7, 13, -3)^T \\
&= (1, 1, -1, 1, 1, -1)^T \\
&\ \ \vdots
\end{aligned}$$

This sequence also converges to a limited cycle. We can thus summarize that the state of second-order BAM may converge to limited cycles.

IV. Lower Bound of Memory Capacity

In the field of associative memories, one of interesting topics is the maximum number of library pairs/patterns that the model can handle such that all the library pairs/patterns are stored as fixed points. The above gives us a non-rigorous sense about the definition of the memory capacity. Without any assumption on the library pairs, from the above definition of the memory capacity, we will easily get some depressing results about the Kosko's encoding method: the memory capacity of BAM with Kosko's encoding method is about 2 or 3 only even for very large p and n (Examples can be found in [7]).

The conventional definition of the memory capacity of associative memories is the maximum number of library patterns (or pairs) that the model can handle such that all library patterns (or pairs) are stored as fixed points with high probability [9]. The assumption on the library pairs is that each component of the library pairs/patterns is a ± 1 equiprobable independent random variable.

Following the conventional definition of the memory capacity (no error in the retrieval pairs), we will show that the memory capacity of second-order BAM is lower bounded by

$$O\left(\min\left(\frac{p^2}{\log p}, \frac{n^2}{\log n}\right)\right).$$

Similar to first order BAM, in second-order BAM a library pair $(\mathbf{X}_h, \mathbf{Y}_h)$ can be retrieved only if it is a fixed point. For the total-order-connection second-order BAM, each library pair is a fixed point if and only if

$$\operatorname{sgn}\left(\sum_{k=1}^{p}\sum_{l=1}^{p} v_{jkl} y_{kh} y_{lh}\right) = x_{jh} \qquad \text{for} \quad j = 1, \cdots, n$$

and

$$\operatorname{sgn}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} u_{kji} x_{jh} x_{ih}\right) = y_{kh} \qquad \text{for} \quad k = 1, \cdots, p.$$

In the partial-order-connection case, each library pair is a fixed point if and only if

$$\operatorname{sgn}\left(\sum_{k=1}^{p-1}\sum_{l>k}^{p} v_{jkl} y_{kh} y_{lh}\right) = x_{jh} \qquad \text{for} \quad j = 1, \cdots, n$$

and

$$\operatorname{sgn}\left(\sum_{j=1}^{n-1}\sum_{i>j}^{n} u_{kji} x_{jh} x_{ih}\right) = y_{kh} \qquad \text{and} \quad k = 1, \cdots, p.$$

In this section, the following assumptions and notation are used.

- $p = rn$, where r is a positive constant.
- Each component of the library pairs $(\mathbf{X}_h, \mathbf{Y}_h)$ is a ± 1 equiprobable independent random variable. The dimensions (n and p) are large.
- For the total-order-connection second-order BAM,

  —EW_{jh} is the event

  $$\operatorname{sgn}\left(\sum_{k=1}^{p}\sum_{l=1}^{p} v_{jkl} y_{kh} y_{lh}\right) = x_{jh}$$

  and $\overline{EW_{jh}}$ is the complement of EW_{jh}.

  —EZ_{kh} is the event

  $$\operatorname{sgn}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} u_{kji} x_{jh} x_{ih}\right) = y_{kh}$$

  and $\overline{EZ_{jh}}$ is the complement of EZ_{kh}.
- Similarly, for the partial-order-connection second-order BAM

  —EW'_{jh} is the event

  $$\operatorname{sgn}\left(\sum_{k=1}^{p-1}\sum_{l>k}^{p} v_{jkl} y_{kh} y_{lh}\right) = x_{jh}$$

  and $\overline{EW'_{jh}}$ is the complement of EW'_{jh}.

  —EZ'_{kh} is the event

  $$\operatorname{sgn}\left(\sum_{j=1}^{n-1}\sum_{i>j}^{n} u_{kji} x_{jh} x_{ih}\right) = y_{kh}$$

  and $\overline{EZ'_{kh}}$ is the complement of EZ'_{kh}.

With the above assumptions, we can deduce the following theorems.

Theorem 1: For the total-order-connection second-order BAM, if the number of library pairs m is less than or equal to

$$\min\left(\frac{n^2}{18\log n}, \frac{p^2}{18\log p}\right)$$

then the probability that each library pair is a fixed point tends to one, as $n \to \infty$ and $p \to \infty$.

Theorem 2: For the partial-order-connection second-order BAM, if the number of library pairs m is less than or equal to

$$\min\left(\frac{n^2}{12\log n}, \frac{p^2}{12\log p}\right)$$

then the probability that each library pair is a fixed point tends to one, as $n \to \infty$ and $p \to \infty$.

From Theorems 1 and 2, the memory capacity of second-order BAM is lower bounded by $O(\min(\frac{n^2}{\log n}, \frac{p^2}{\log p}))$.

Proof of Theorem 1:

Lemma 1: The probability $\text{Prob}(\overline{EW_{jh}})$ is

$$Q\left(\frac{p^2}{\sqrt{(m-1)(3p^2-2p)}}\right)$$

for $j = 1, \cdots, n$ and $h = 1, \cdots, m$. Also, $Q(z)$ is defined as

$$Q(z) = \frac{1}{\sqrt{2\pi}}\int_{z}^{\infty} \exp\left(\frac{-z^2}{2}\right) dz.$$

Proof: We define

$$w_{jh} = \sum_{k=1}^{p}\sum_{l=1}^{p} v_{jkl} y_{kh} y_{lh}. \tag{13}$$

Without loss of generality, we consider the library pair $(\mathbf{X}_h, \mathbf{Y}_h)$ having all components positive: $\mathbf{X}_h = (1, \cdots, 1)^T$ and $\mathbf{Y}_h = (1, \cdots, 1)^T$ (This consideration is usually used [12, 14] and does not affect our results. We can easily verify this by use of conditional probability.). Substituting (6) into (13), w_{jh} becomes

$$\begin{aligned} w_{jh} &= p^2 + \sum_{h' \neq h}^{m} x_{jh'} \sum_{k=1}^{p}\sum_{l=1}^{p} y_{kh'} y_{lh'} \\ &= p^2 + N. \end{aligned}$$

Clearly, N is a sum of $(m-1)$ identically independent random variables. Also,

$$E\left[x_{jh'}\sum_{k=1}^{p}\sum_{l=1}^{p} y_{kh'} y_{lh'}\right] = \left[x_{jh'}\left(\sum_{k=1}^{p} y_{kh'}\right)^2\right] = 0$$

and

$$E\left[(x_{jh'})^2\left(\sum_{k=1}^{p} y_{kh'}\right)^4\right] = 3p^2 - 2p$$

where $E[\cdot]$ is the expectation operator. Hence, N is a random variable with zero mean and variance $(m-1)(3p^2-2p)$. According to central limit theorem, for large m the distribution of $\frac{N}{\sqrt{(m-1)(3p^2-2p)}}$ approaches standard normal. Then, the probability that $w_{jh} < 0$ is

$$Q\left(\frac{p^2}{\sqrt{(m-1)(3p^2-2p)}}\right).$$

Hence, $\mathrm{Prob}(\overline{EW_{jh}})$ is

$$Q\left(\frac{p^2}{\sqrt{(m-1)(3p^2-2p)}}\right).$$

for $j=1,\cdots,p$ and $h=1,\cdots,m$. Note that in the proof we neglect the case $w_{jh}=0$. For large m, the distribution of $\frac{N}{\sqrt{(m-1)(3p^2-2p)}}$ approaches standard normal. Thus, neglecting the case will not affect the result. For a standard normal density function,

$$\frac{1}{\sqrt{2\pi}}\int_{z'\to z}^{\infty}\exp\left(\frac{-x^2}{2}\right)dx = \frac{1}{\sqrt{2\pi}}\int_{z}^{\infty}\exp\left(\frac{-x^2}{2}\right)dx.$$

Q.E.D.

Similarly, Lemma 2 can be obtained.

Lemma 2: The probability $\mathrm{Prob}(\overline{EZ_{kh}})$ is

$$Q\left(\frac{n^2}{\sqrt{(m-1)(3n^2-2n)}}\right)$$

for $j=1,\cdots,n$ and $k=1,\cdots,m$.

Let the probability that all library pairs are fixed points be P_*

$$\begin{aligned}P_* &= \mathrm{Prob}(EW_{11}\cap EW_{21}\cap\cdots\cap EW_{nm}\\ &\qquad\cap EZ_{11}\cap EZ_{21}\cap\cdots\cap EZ_{pm})\\ &= 1-\mathrm{Prob}(\overline{EW_{11}}\cup\overline{EW_{21}}\cup\cdots\cup\overline{EW_{nm}}\\ &\qquad\cup\overline{EZ_{11}}\cup\overline{EZ_{21}}\cup\cdots\cup\overline{EZpm})\\ &\geq 1-mn\mathrm{Prob}(\overline{EW_{11}})-mp\mathrm{Prob}(\overline{EZ_{11}}).\end{aligned} \tag{14}$$

Note that

$$P_* \neq (\mathrm{Prob}(EW_{11}))^{mn}(\mathrm{Prob}(EZ_{11}))^{mp} \tag{15}$$

because the events EW_{jh}'s and EZ_{kh}'s are not mutually independent. That can be easily observed when $m=2$.

From the inequalities

$$Q\left(\frac{n^2}{\sqrt{(m-1)(3n^2-2n)}}\right) < Q\left(\frac{n}{\sqrt{3m}}\right)$$

and

$$Q\left(\frac{p^2}{\sqrt{(m-1)(3p^2-2p)}}\right) < Q\left(\frac{p}{\sqrt{3m}}\right)$$

and the lemmas, (14) becomes

$$P_* \geq 1-mnQ\left(\frac{p}{\sqrt{3m}}\right)-mpQ\left(\frac{n}{\sqrt{3m}}\right). \tag{16}$$

Letting $P_B = mnQ(\frac{p}{\sqrt{3m}})$ and $P_A = mpQ(\frac{n}{\sqrt{3m}})$, we get

$$P_* \geq 1-P_B-P_A. \tag{17}$$

If z is large [11]

$$Q(z)\approx\exp\left\{-\frac{z^2}{2}-\log z-\frac{1}{2}\log 2\pi\right\} \tag{18}$$

which is quite accurate for $z>3$. Using the approximation (18)

$$\begin{aligned}P_A &= \exp\left\{\log m+\log p-\frac{n^2}{6m}-\log\frac{n}{\sqrt{3m}}-\frac{1}{2}\log 2\pi\right\}\\ &= \exp\left\{\frac{3}{2}\log m-\frac{n^2}{6m}+\log r+\frac{1}{2}\log 3-\frac{1}{2}\log 2\pi\right\}.\end{aligned} \tag{19}$$

Considering the first two terms, if we set $m=\frac{n^2}{18\log n}$, P_A will become

$$\exp\left\{-\frac{3}{2}\log(18\log n)+\text{constants}\right\}.$$

Moreover, as $n\to\infty$, $P_A\to 0$. Since, P_A is an increasing function of m, we can conclude that as $n\to\infty$ and $m\leq\frac{n^2}{18\log n}$, $P_A\to 0$. Similarly, we can get that as $p\to\infty$ and $m\leq\frac{p^2}{18\log p}$, $P_B\to 0$. To sum up, for large n and p, if $m\leq\min(\frac{n^2}{18\log n},\frac{p^2}{18\log p})$, then $P_*\to 1$ (end of the proof of Theorem 1).

Remark: Equations (14) and (15) mark the difference between our approach in this paper and the approach in [15]. Amari *et al.* [15] neglected the dependence among the random variables. If we directly use the approach in [15], we will get the following *wrong statement*

$$P_* = (\mathrm{Prob}(EW_{11}))^{mn}(\mathrm{Prob}(EZ_{11}))^{mp}$$

Here, we use (14) and obtain a lower bound of the capacity **only** instead of the actual capacity.

Proof of Theorem 2:

Lemma 3: The probability $\mathrm{Prob}(\overline{EW'_{jh}})$ is

$$Q\left(\sqrt{\frac{p(p-1)}{2(m-1)}}\right)$$

for $j=1,\cdots,n$ and $k=1,\cdots,m$.

Proof: We define

$$w'_{jh}=\sum_{k=1}^{p-1}\sum_{l>k}^{p}v_{jkl}y_{kh}y_{lh}. \tag{20}$$

Similar to Lemma 1, we assume that the library pair $(\mathbf{X}_h,\mathbf{Y}_h)$ has positive components all. Substituting (10) into (20), w'_{jh} becomes

$$\begin{aligned}w'_{jh} &= \frac{p(p-1)}{2}+\sum_{h'\neq h}^{m}x_{jh'}\sum_{k=1}^{p-1}\sum_{l>k}^{p}y_{kh'}y_{lh'}\\ &= \frac{p(p-1)}{2}+N'.\end{aligned}$$

N' is a sum of $(m-1)$ identically independent random variables. Also

$$E\left[x_{jh'}\sum_{k=1}^{p-1}\sum_{l>k}^{p}y_{kh'}y_{lh'}\right]=0$$

and

$$E\left[(x_{jh'})^2\left(\sum_{k=1}^{p}\sum_{l>k}^{p-1}y_{kh'}y_{lh'}\right)^2\right]=\frac{p(p-1)}{2}.$$

Hence, N' is a random variable with zero mean and variance $\frac{(m-1)p(p-1)}{2}$. For large m, the distribution of the normalized N'

$$\frac{N'}{\sqrt{\frac{(m-1)p(p-1)}{2}}}$$

approaches standard normal. It then follows that the probability that $w'_{jh} < 0$ is

$$Q\left(\sqrt{\frac{p(p-1)}{2(m-1)}}\right).$$

Hence, $\mathrm{Prob}(\overline{EW'_{jh}})$ is

$$Q\left(\sqrt{\frac{p(p-1)}{2(m-1)}}\right)$$

for $j = 1, \cdots, p$ and $h = 1, \cdots, m$. **Q.E.D.**

Similarly, one can easily get Lemma 4.

Lemma 4: The probability $\mathrm{Prob}(\overline{EZ'_{kh}})$ that $\mathrm{sgn}(\sum_{j=1}^{n}\sum_{i>j}^{n} u_{kji}x_{jh}x_{ih}) \neq y_{kh}$ is

$$Q\left(\sqrt{\frac{n(n-1)}{2(m-1)}}\right)$$

for $j = k, \cdots, p$ and $h = 1, \cdots, m$.

Define P'_* be the probability that all library are fixed points for the partial connection second-order BAM. Then

$$P'_* \geq 1 - mn\mathrm{Prob}(\overline{EW'_{11}}) - mp\mathrm{Prob}(\overline{EZ'_{11}}). \tag{21}$$

With Lemmas 3 and 4, (21) becomes

$$\begin{aligned} P'_* &\geq 1 - mnQ\left(\sqrt{\frac{p(p-1)}{2(m-1)}}\right) - mpQ\left(\sqrt{\frac{n(n-1)}{2(m-1)}}\right) \\ &\geq 1 - mnQ\left(\sqrt{\frac{p(p-1)}{2m}}\right) - mpQ\left(\sqrt{\frac{n(n-1)}{2m}}\right). \end{aligned} \tag{22}$$

Define $P'_B = mnQ(\sqrt{\frac{p(p-1)}{2m}})$ and $P'_A = mpQ(\sqrt{\frac{n(n-1)}{2m}})$. Then

$$P'_* \geq 1 - P'_B - P'_A. \tag{23}$$

Similar to the proof of Theorem 1, we can also prove that if $m \leq \min(\frac{n^2}{12\log n}, \frac{p^2}{12\log p})$, then $P'_* \to 1$ as n and p tends ∞. (end of the proof of Theorem 2).

V. Confidence Dynamics

It is known that if small number of errors is allowed in the retrieval pattern, the memory capacity of the higher-order Hopfield network can be proportional αn^g, where g is the order of the network and n is the number of neurons [12]. However, the results of the higher-order Hopfield network in [12] is based on the stabilization of the network during recalling. Since the stabilization of second-order BAM is not guaranteed, we cannot use the approach in [12] to estimate its memory capacity when small number of errors is allowed in the retrieval pairs.

In this section, we present the statistical dynamics of second-order BAM: Starting with an initial state close to the library pairs (there are some errors in the initial state), how the confidence interval of the number of errors changes during recalling. If the confidence interval converges to a small value, then the library pair can be recalled with small number of errors. The final value of the confidence interval represents the upper bound of the number of errors in the retrieval pairs. The maximum number of errors in the initial state such that the confidence interval converges to a small value represents the lower bound of attraction basin of the library pairs. Also, the maximum number of library pairs such that the confidence interval of the number of errors converges to a very small value represents the lower bound of the memory capacity.

Definition 1: Given that $p = rn$ and $m = \alpha n^2$, P_Y^{**} is the probability that the fraction of errors in F_Y in the next state is less than ρ_y (i.e. the Hamming distance between $\mathbf{Y}_h$ and $\mathbf{Y}^{(t)}$ is less than $\rho_y p$), for every library pair $(\mathbf{X}_h, \mathbf{Y}_h)$ and for any $\rho_x^{(t)} n$ errors in F_X in the present state (i.e., the Hamming distance between $\mathbf{X}_h$ and $\mathbf{X}^{(t)}$ is equal to $\rho_x^{(t)} n$).

We first estimate a lower bound of P_Y^{**}. Then we can find out the minimum value of ρ_y, denoted as ρ_y^*, such that P_Y^{**} tends to one. The above means that given $\rho_x^{(t)}$, the probability that the fraction of errors in F_Y in the next state is less than ρ_y^* tends to one.

Definition 2: Given that $p = rn$ and $m = \alpha n^2$, P_X^{**} is the probability that the fraction of errors in F_X in the next state is less than ρ_x, for every library pair $(\mathbf{X}_h, \mathbf{Y}_h)$ and for any $\rho_y^{(t+1)} p$ errors in F_Y in the present state.

Also, we can find out the minimum value of ρ_x, denoted as ρ_x^*, such that P_X^{**} tends to one. From ρ_x^* and ρ_y^*, we can construct a dynamic curve about the confidence interval of the fraction of errors. One can easily use the above dynamics to predict the memory capacity, attraction basin, and final error in the retrieval pairs when error in the retrieval pairs is allowed. The notations and assumptions used here are:

- $p = rn$, where r is a positive constant.
- Each component of the library pairs $(\mathbf{X}_h, \mathbf{Y}_h)$ is a ± 1 equiprobable independent random variable. The dimensions (n and p) are large.
- $m = \alpha n^2$, where α is a positive constant.
- $EA_{h,g}$ is the event

$$d(\mathbf{Y}^{(t+1)}, \mathbf{Y}_h) < \rho_y p$$

for a given library pair $(\mathbf{X}_h, \mathbf{Y}_h)$ and a given present state $\mathbf{X}^{(t)}$ which is an element of the set

$$S_{h,t} = \{\mathbf{X} \in \{+1, -1\}^n \text{ such that } d(\mathbf{X}, \mathbf{X}_h) = \rho_x^{(t)} n\}$$

where $d(\cdot, \cdot)$ is the Hamming distance between two bipolar vectors. Note that the number of elements in the set $S_{h,t}$ is $\binom{n}{\rho_x^{(t)} n}$. Thus the index g has the range from 1 to $\binom{n}{\rho_x^{(t)} n}$. Also, $\overline{EA_{h,g}}$ is the complement event of $EA_{h,g}$

$$d(\mathbf{Y}^{(t+1)}, \mathbf{Y}_h) \geq \rho_y p.$$

- EA is the event that

$$d(\mathbf{Y}^{(t+1)}, \mathbf{Y}_h) < \rho_y p.$$

for every library pair $(\mathbf{X}_h, \mathbf{Y}_h)$ and every $\mathbf{X}^{(t)} \in S_{h,t}$, Also, $\overline{EA}$ is the complement event of EA. Hence

$$\overline{EA} = \bigcup_{h,g} \overline{EA_{h,g}}.$$

and

$$P_Y^{**} \equiv \mathrm{Prob}(EA).$$

Lemma 5: For large n and p (i.e. $n \to \infty$, $p \to \infty$)

$$\mathrm{Prob}(\overline{EA_{g,h}}) \leq \exp\left\{rn\hbar(\rho_y) - \frac{\rho_y r(n-1)(1 - 2\rho_x^{(t)})^4}{c\alpha}\right\}$$

where

$$\hbar(\rho) = -\rho\log\rho - (1-\rho)\log(1-\rho)$$

for $g = 1, \cdots, \binom{n}{\rho_x^{(t)} n}$ and $h = 1, \cdots, m$. For total connection second-order BAM, c is 6. For partial connection second-order BAM, c is 4.

Proof: Without loss of generality, we consider the library pair $(\mathbf{X}_h, \mathbf{Y}_h)$ having all components positive: $\mathbf{X}_h = (1, \cdots, 1)^T$ and $\mathbf{Y}_h = (1, \cdots, 1)^T$. Let J be the set of indices in which $\mathbf{X}^{(t)}$ and $\mathbf{X}_h$ differ. For a given $\mathbf{X}^{(t)} \in S$, there is only one J and $|J| = \rho_x^{(t)} n$. Also, let K be the set of indices of $\mathbf{Y}_h$ and $\mathbf{Y}^{(t)}$ such that $|K| = \rho_y p$. Note that there are $\binom{p}{\rho_y p}$ such sets of K.

For the total-order-connection second-order BAM, event $\overline{EA_{g,h}}$ implies that there are at least one K, where $|K| = \rho_y p$, such that

$$\sum_{k\in K}\sum_{j=1}^{n}\sum_{i=1}^{n} u_{kji} x_i^{(t)} x_j^{(t)} < 0. \tag{24}$$

Hence, we have

$$\begin{aligned}
&\mathrm{Prob}(\overline{EA_{g,h}})\\
&\quad \le \mathrm{Prob}\Bigg(\text{there are at least one } K \text{ where } |K| = \rho_y p\\
&\qquad\qquad \text{such that } \sum_{k\in K}\sum_{j=1}^{n}\sum_{i=1}^{n} u_{kji} x_i^{(t)} x_j^{(t)} < 0\Bigg)\\
&\quad \le \binom{p}{\rho_y p}\mathrm{Prob}\Bigg(\sum_{k\in K}\sum_{j=1}^{n}\sum_{i=1}^{n} u_{kji} x_i^{(t)} x_j^{(t)} < 0\\
&\qquad\qquad \text{for a given } K\Bigg).
\end{aligned} \tag{25}$$

Let

$$P'' = \mathrm{Prob}\left(\sum_{k\in K}\sum_{j=1}^{n}\sum_{i=1}^{n} u_{kji} x_i^{(t)} x_j^{(t)} < 0 \text{ for a given } K\right). \tag{26}$$

According to the assumption of the library pair $(\mathbf{X}_h, \mathbf{Y}_h)$ having all components positive

$$\begin{aligned}
P'' = \mathrm{Prob}\Bigg(&\rho_y p\left(1 - 2\rho_x^{(t)}\right)^2 n^2\\
&+ \sum_{h'\ne h}^{m}\sum_{k\in K} y_{kh'}\left(\sum_{j\notin J} x_{jh'} - \sum_{j\in J} x_{jh'}\right)^2 < 0\Bigg).
\end{aligned} \tag{27}$$

One can easily prove that

$$E\left[\sum_{k\in K} y_{kh'}\left(\sum_{j\notin J}^{n} x_{jh'} - \sum_{j\in J} x_{jh'}\right)^2\right] = 0$$

and

$$E\left[\left(\sum_{k\in K} y_{kh'}\left(\sum_{j\notin J}^{n} x_{jh'} - \sum_{j\in J} x_{jh'}\right)\right)^2\right] = \rho_y p(3n^2 - 2n).$$

As $\sum_{k\in K} y_{kh'}(\sum_{j\notin J}^{n} x_{jh'} - \sum_{j\in J} x_{jh'})^2$'s are identically independent random variables, for large n and p (i.e. m is large due to $m = \alpha n^2$) we can apply central limit theorem to get

$$\begin{aligned}
P'' &= Q\left(\frac{\rho_y p\left(1 - 2\rho_x^{(t)}\right)^2 n^2}{\sqrt{\rho_y p(m-1)(3n^2 - 2n)}}\right)\\
&\le Q\left(\frac{\rho_y r n\left(1 - 2\rho_x^{(t)}\right)^2 n^2}{\sqrt{\rho_y r n \alpha n^2 3n^2}}\right).
\end{aligned}$$

Using the approximation (18),

$$\begin{aligned}
P'' &\le \exp\left\{-\frac{\rho_y r n\left(1 - 2\rho_x^{(t)}\right)^4}{6\alpha}\right\}\\
&\le \exp\left\{-\frac{\rho_y r (n-1)\left(1 - 2\rho_x^{(t)}\right)^4}{6\alpha}\right\}.
\end{aligned} \tag{28}$$

Note that the purpose of changing n to $(n-1)$ in (28) is to unify the result of total connection second-order and that of partial connection second-order BAM.

Then

$$\begin{aligned}
&\mathrm{Prob}(\overline{EA_{g,h}})\\
&\quad \le \binom{p}{\rho_y p}\exp\left\{-\frac{\rho_y r (n-1)\left(1 - 2\rho_x^{(t)}\right)^4}{6\alpha}\right\}.
\end{aligned} \tag{29}$$

By a standard use of Stirling's asymptotic formula for factorial, one can find that the binomial coefficient

$$\binom{p}{\rho_y p} \sim \exp\{p\hbar(\rho_y)\}$$

if p is large and ρ_y is constant between 0 to 1.

Replacing the binomial coefficient in (29) with Stirling's asymptotic formula, for the total-order-connection second-order BAM

$$\mathrm{Prob}(\overline{EA_{g,h}}) \le \exp\left\{rn\hbar(\rho_y) - \frac{\rho_y r (n-1)\left(1 - 2\rho_x^{(t)}\right)^4}{6\alpha}\right\}.$$

In the partial-order-connection case, event $\overline{EA_{g,h}}$ implies that there is at least one K, where $|K| = \rho_y p$, such that

$$\sum_{k\in K}\sum_{j=1}^{n-1}\sum_{i>j}^{n} u_{kji} x_i^{(t)} x_j^{(t)} < 0. \tag{30}$$

Then

$$\begin{aligned}
&\mathrm{Prob}(\overline{EA_{g,h}})\\
&\quad \le \binom{p}{\rho_y p}\mathrm{Prob}\left\{\sum_{k\in K}\sum_{j=1}^{n-1}\sum_{i>j}^{n} u_{kji} x_i^{(t)} x_j^{(t)} < 0 \text{ for a given } K\right\}.
\end{aligned} \tag{31}$$

Similar to total-order-connection case, let

$$\begin{aligned}
P'' &= \mathrm{Prob}\left\{\sum_{k\in K}\sum_{j=1}^{n}\sum_{i>j}^{n} u_{kji} x_i^{(t)} x_j^{(t)} < 0 \text{ for a given } K\right\}\\
&= \mathrm{Prob}\Bigg\{\sum_{h'=1}^{m}\sum_{k\in K} y_{kh'}\Bigg(\sum_{j\notin J, i\notin J, i>j} x_{ih'} x_{jh'} x_i^{(t)} x_j^{(t)}\\
&\qquad + \sum_{j\in J, i\in J, i>j} x_{ih'} x_{jh'} x_i^{(t)} x_j^{(t)}\\
&\qquad + \sum_{j\in J, i\notin J, i>j} x_{ih'} x_{jh'} x_i^{(t)} x_j^{(t)}\\
&\qquad + \sum_{j\notin J, i\in J, i>j} x_{ih'} x_{jh'} x_i^{(t)} x_j^{(t)}\Bigg) < 0\Bigg\}\\
&= \mathrm{Prob}\Bigg\{\sum_{h'=1}^{m}\sum_{k\in K} y_{kh'}\Bigg(\sum_{j\notin J, i\notin J, i>j} x_{ih'} x_{jh'} x_i^{(t)} x_j^{(t)}\\
&\qquad + \sum_{j\in J, i\in J, i>j} x_{ih'} x_{jh'} x_i^{(t)} x_j^{(t)}\\
&\qquad + \sum_{j\notin J, i\in J} x_{ih'} x_{jh'} x_i^{(t)} x_j^{(t)}\Bigg) < 0\Bigg\}\\
&= \mathrm{Prob}\Bigg\{\rho_y p\Bigg(\frac{\left(1 - \rho_x^{(t)}\right)n\left(\left(1 - \rho_x^{(t)}\right)n - 1\right)}{2}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\rho_x^{(t)}n)(\rho_x^{(t)}n-1)}{2}-(1-\rho_x^{(t)})(\rho_x^{(t)})n^2\Bigg)\\
&+\sum_{h'\neq h}\sum_{k\in K} y_{kh'}\Bigg(\sum_{j\notin J, i\notin J, i>j} x_{ih'}x_{jh'}\\
&+\sum_{j\in J, i\in J, i>j} x_{ih'}x_{jh'}-\sum_{j\notin J, i\in J} x_{ih'}x_{jh'}\Bigg)<0\Bigg\}\\
&=\mathrm{Prob}\Bigg\{\rho_y p\Bigg(\frac{n(n-1)}{2}-(1-\rho_x^{(t)})(\rho_x^{(t)})n^2\Bigg)\\
&+\sum_{h'\neq h}\chi_{h'}<0\Bigg\}. \qquad (32)
\end{aligned}
$$

Also

$$E[\chi_{h'}]=0, \quad \text{and} \quad E[\chi_{h'}^2]=\rho_y p\frac{n(n-1)}{2}.$$

Applying the central limit theorem,

$$
\begin{aligned}
P''&=Q\left(\frac{\rho_y p\left(\frac{n(n-1)}{2}-(1-\rho_x^{(t)})(\rho_x^{(t)})n^2\right)}{\sqrt{\rho_y p(m-1)\frac{n(n-1)}{2}}}\right)\\
&\le Q\left(\frac{\frac{\rho_y r n\left(1-2\rho_x^{(t)}\right)^2 n(n-1)}{2}}{\sqrt{\rho_y r n\alpha n^2\frac{n(n-1)}{2}}}\right). \qquad (33)
\end{aligned}
$$

Thus, for the partial-order-connection second-order BAM

$$
\begin{aligned}
&\mathrm{Prob}(\overline{EA_{g,h}})\\
&\quad\le \exp\left\{rn\hbar(\rho_y)-\frac{\rho_y r(n-1)(1-2\rho_x^{(t)})^4}{4\alpha}\right\}.
\end{aligned}
$$

Q.E.D.

From the definition of the event EA

$$
\begin{aligned}
P_Y^{**}&\equiv \mathrm{Prob}(EA)=1-\mathrm{Prob}(\overline{EA})\\
&\ge 1-\sum_{h=1}^{m}\sum_{g=1}^{\binom{n}{\rho_x^{(t)}n}}\mathrm{Prob}(\overline{EA_{g,h}})\\
&=1-m\binom{n}{\rho_x^{(t)}n}\mathrm{Prob}(\overline{EA_{1,1}}). \qquad (34)
\end{aligned}
$$

With Lemma 5, we can immediately have the following theorem.

Theorem 3: For large n and p

$$
\begin{aligned}
P_Y^{**}\ge 1-\exp\Bigg\{&n\hbar(\rho_x^{(t)})+\log\alpha n^2+rn\hbar(\rho_y)\\
&-\frac{\rho_y r(n-1)(1-2\rho_x^{(t)})^4}{c\alpha}\Bigg\}.
\end{aligned}
$$

For total connection second-order BAM, c is 6. For partial connection second-order BAM, c is 4.

Let ρ_y^* be the minimum value of ρ_y such that $P_Y^{**}\to 1$ as $n\to\infty$ and $p\to\infty$. From Theorem 3, ρ_y^* is the minimum value of ρ_y such that

$$\hbar(\rho_x^{(t)})+r\hbar(\rho_y)-\frac{\rho_y r(1-2\rho_x^{(t)})^4}{c\alpha}<0. \qquad (35)$$

Apparently, for a given $\rho_x^{(t)}\in[0,0.5)$, ρ_y^* can be solved graphically as in Fig. 1. Let ρ_y' be the intersect of the line

$$L_1: y=\frac{\rho_y(1-2\rho_x^{(t)})^4}{c\alpha}-\frac{\hbar(\rho_x^{(t)})}{r} \qquad (36)$$

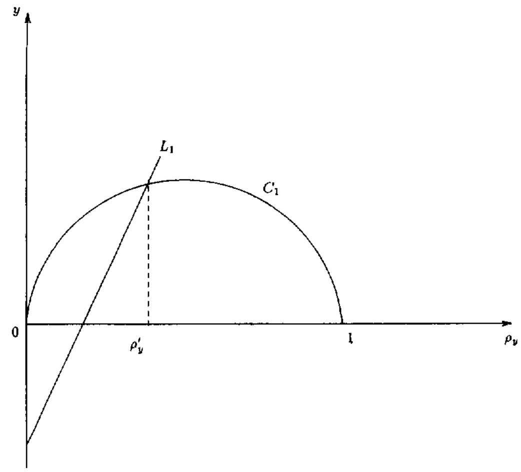

Fig. 1. Graphical implication of solving ρ_y'. The next state ρ_y' can be solved by finding out the intersection of C_1 and L_1.

and the curve

$$C_1: y=\hbar(\rho_y). \qquad (37)$$

Then

$$\rho_y^*=\rho_y'+\varepsilon$$

where ε is an arbitrary small positive constant.

Note that $\hbar(\rho_x^{(t)})$ is an increasing function of $\rho_x^{(t)}\in[0,5)$ and $(1-2\rho_x^{(t)})^4$ is a decreasing function of $\rho_x^{(t)}\in[0,5)$. From (36) and Fig. 1, for a smaller $\rho_x^{(t)}\in(0,0.5)$ (Thus, the line is shifted up and the slope of the line increases.), a smaller ρ_y^* can be obtained. Thus, the following corollary can be obtained.

Corollary 1: $\rho_{x1}^{(t)}<\rho_{x2}^{(t)}<0.5$ implies that $\rho_{y1}'<\rho_{y2}'$.

From Theorem 3 and Corollary 1, as $n\to\infty$ and $p\to\infty$, for every library pair $(\mathbf{X}_h,\mathbf{Y}_h)$ and $d(\mathbf{X}_h,\mathbf{X}^{(t)})\le\rho_x^{(t)}n$ (i.e. the fraction of error in the present state of F_X is less than or equal to $\rho_x^{(t)}$), where $\rho_x^{(t)}\in[0,5)$, the probability that the fraction of error in the next states of F_Y is less than $\rho_y'+\varepsilon$ tends to one. As ε can be any arbitrary small positive constant, one can restate the above statement as:

Corollary 2: As $n\to\infty$ and $p\to\infty$, for every library pair $(\mathbf{X}_h,\mathbf{Y}_h)$ and every $\mathbf{X}^{(t)}$ such that $d(\mathbf{X}_h,\mathbf{X}^{(t)})\le\rho_x^{(t)}n$ ($\rho_x^{(t)}<0.5$), the probability that $d(\mathbf{Y}_h,\mathbf{Y}^{(t+1)})\le\rho_y'$ tends to one, where ρ_y' is the intersect of L_1 and C_1 as shown in Fig. 1.

The above corollary means that if the fraction of error in the present state of F_X being less than or equal to $\rho_x^{(t)}$, then the fraction of error in the next states of F_Y is less than or equal to ρ_y' (denoted as $\rho_y^{(t+1)}$). Using similar method, one can get that

Theorem 4: For large n and p,

$$
\begin{aligned}
P_X^{**}\ge 1-\exp\Bigg\{&p\hbar(\rho_y^{(t+1)})+\log\alpha\left(\frac{p}{r}\right)^2+\frac{p}{r}\hbar(\rho_x)\\
&-\frac{\rho_x r(p-1)(1-2\rho_y^{(t+1)})^4}{c\alpha}\Bigg\}.
\end{aligned}
$$

For total connection second-order BAM, c is 6. For partial connection second-order BAM,c is 4.

Let ρ_x^* be the minimum value of ρ_x such that P_X^{**} tends to one. Also, one can find out ρ_x^* and ρ_x' by considering the intersect of the line

$$L_2: y=\frac{\rho_x r^2(1-2\rho_x^{(t)})^4}{c\alpha}-r\hbar(\rho_y^{(t+1)}) \qquad (38)$$

and the curve

$$C_2 : y = \hbar(\rho_x). \tag{39}$$

Then Corollary 3 and 4 are obtained.

Corollary 3: $\rho_{y1}^{(t+1)} < \rho_{y2}^{(t+1)} < 0.5$ implies that $\rho'_{x1} < \rho'_{x2}$.

Corollary 4: As $n \to \infty$ and $p \to \infty$, for every library pair $(\mathbf{X}_h, \mathbf{Y}_h)$ and every $\mathbf{Y}^{(t+1)}$ such that $d(\mathbf{Y}_h, \mathbf{Y}^{(t+1)}) \leq \rho_y^{(t+1)}$ ($\rho_y^{(t+1)} < 0.5$), the probability that $d(\mathbf{X}_h, \mathbf{X}^{(t+1)}) \leq \rho'_x$ tends to one, where ρ'_x is the intersect of L_2 and C_2.

That means that if the fraction of error in the present state of F_Y being less than or equal to $\rho_y^{(t+1)}$, then the fraction of error in the next states of F_X is less than or equal to ρ'_x (denoted as $\rho_x^{(t+1)}$). By iteratively solving $\rho_x^{(t+1)}$ and $\rho_y^{(t+1)}$ with a given initial fraction of errors being zero, we can construct two sequences, $\rho_x^{(t)}$ and $\rho_y^{(t)}$. These sequences are the statistical dynamics about the confidence interval of the fraction of errors. We can easily use the dynamics to estimate the memory capacity (errors are allowed in the retrieval pairs), the attraction basin, and the number of errors in the retrieval pairs.

For example, if $\alpha = \alpha'$ and initial fractions of errors ($\rho_x^{(0)} = \rho_{\text{maxinit}}$) being nonzero, if the sequences converge to the small value ($\rho_x^{\text{final}}, \rho_y^{\text{final}}$) which are less than $\rho_x^{(0)}$, then the memory capacity is at least equal to $\alpha' n$ and the attraction region of all library pairs is at least equal to $\rho_{\text{maxinit}} n$. Also, ρ_x^{final} and ρ_y^{final} reflect the upper bound of the number of errors in the retrieval pairs. In the following, we will use several numerical examples to illustrate how to estimate the memory capacity, the attraction basin, the final error in the retrieval pairs of the partial connection second-order BAM. That of total connection second-order BAM can also be studied in the same way.

Numerical Example a: Here, we study the dynamics of the partial connection second-order BAM. The dynamics about ($\rho_x^{(t)}, \rho_y^{(t)}$) for ($\alpha = 0.01$ and $r = 1$) and ($\alpha = 0.02$ and $r = 2$) are constructed. In Figs. 2 and 3, the sequences ($\rho_x^{(t)}, \rho_y^{(t)}$) with different initial conditions $\rho_x^{(0)} = 0$ and 0.05 are plotted. From Fig. 2, we can conclude that the memory capacity of partial connection second-order BAM is at least equal to $0.01n^2$ for $r = 1$. Also, the corresponding attraction basin is at least equal to $0.05n$ and the final error in the retrieval pairs is less than or equal to $1.02 \times 10^{-5} n$ for $r = 1$. From Fig. 3, the memory capacity of partial connection second-order BAM is at least equal to $0.02n^2$ for $r = 2$. Also, the corresponding attraction basin is at least equal to $0.05n$ and the final errors in the retrieval pairs are less than or equal to $7.14 \times 10^{-5} n$ in F_X and $1.43 \times 10^{-4} p$ in F_Y.

Numerical Example b: In this example, we study the memory capacity of partial connection second-order BAM for different values of r. For a given r, let α' be the largest value of α such that the sequences with a large initial $\rho_x^{(0)}$ converge to small ($\rho_x^{\text{final}}, \rho_y^{\text{final}}$). Then $\alpha' n^2$ can be considered as a lower bound of the memory capacity of partial connection second-order BAM. When we use the method of example a, Table I which summarizes the α''s for different values of r is obtained. From the table, the memory capacity increases with r. Also, there are some symmetrical results about α'. For example, when $r = 0.2$, α' is 0.002 14. If we divide this value by 0.2^2, the new value is similar to α' at $r = 5$. One can easily verify other cases by interchanging p and n, and putting $r' = \frac{1}{r}$.

Numerical Example c: In example b, we do not show ρ_{maxinit}, ρ_x^{final} and ρ_y^{final} when $\alpha = \alpha'$. It is because ρ_{maxinit} is only a little bit more than ρ_x^{final} when the number of pair is equal to the lower bound $\alpha' n^2$.

In this example, we will study the attraction basin while $r = 1$ and $r = 2$. For a given α, let $\rho_{\text{maxinit},r=k}$ be the largest values of $\rho_x^{(0)}$ such that the sequences converge. Tables II and III give us the summary of the attraction basin at different values of α. When

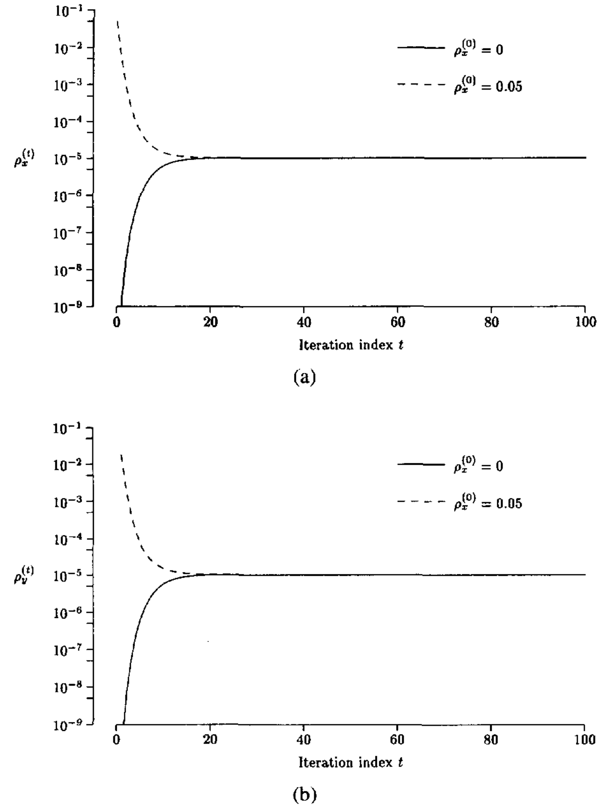

Fig. 2. The confidence dynamics of partial connection second-order BAM with two initial conditions ($\rho_x^{(0)} = 0, 0.05$) for $\alpha = 0.01$ and $r = 1$. (a) The dynamics of $\rho(t)$. (b) The dynamics of $\rho_y(t)$. Since all sequences converge, the attraction basin is at least equal to $0.05n$.

$r = 1$, ρ_y^{final} is same as ρ_x^{final}. When the number of pair is decreased, the attraction basin is increased and the final error in the retrieval is decreased.

Remark: In [14] and [15], based on the law of large number [16], Amari has studied the dynamic behavior of first-order Hopfield network. The considerations of dynamics are "for a library pattern" and "for a error pattern." However, in this paper the higher-order BAM is studied. Extension of general high-order is discussed in the next section. According to (34), the considerations here are "for every library pair" and "for every error pattern." Thus, the dynamics presented here is under stronger conditions than that in [14], [15].

In [13], a more rigorous dynamic behavior of first-order Hopfield network has been studied based on the "energy landscape" (see back Section 4.1 in [13]). Also, in [12] when error in the retrieval pairs is allowed, the memory capacity of general-order Hopfield network has been studied based on the "energy landscape" (see back Section 2 in [12]). Unfortunately, the state of second-order BAM may converge to limited cycles. It means that the method of "energy landscape" cannot be used here. Moreover, when our formulation of P^{**} is used, one can solve ρ'_x and ρ'_y by a simple numerical method. Thus, we can easily estimate the capacity, attraction basin, and final error for different value of r.

On the other hand, according to our method, whenever $\rho_x^{(t)} > 0.5$ or $\rho_y^{(t)} > 0.5$, the iteration must be stopped. No conclusion can be made.

VI. Extension to Higher-Order BAM

Although our results are mainly about second-order BAM, one can apply similar method to analyze the higher-order BAM or some associative memories with the limited cycle behavior. For example, in

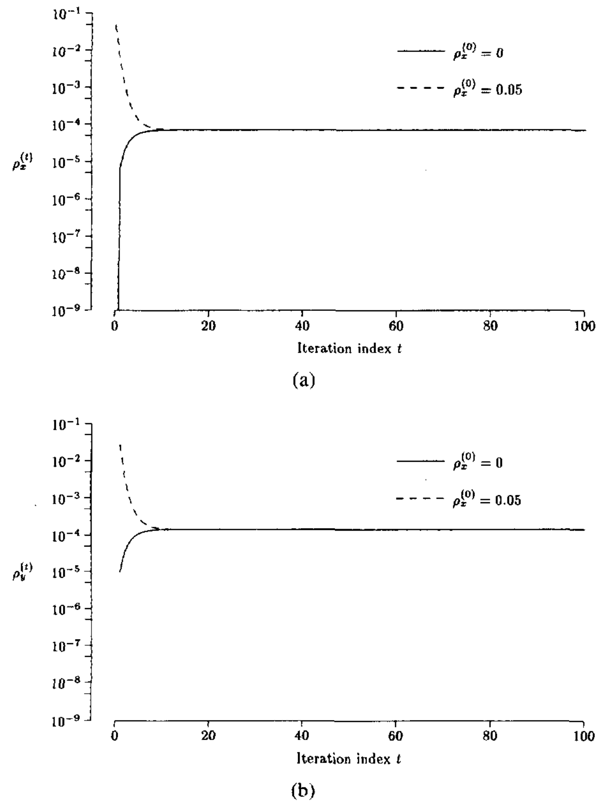

Fig. 3. The confidence dynamics of partial connection second-order BAM with two initial conditions ($\rho_x^{(0)} = 0, 0.05$) for $\alpha = 0.02$ and $r = 2$. (a) The dynamics of $\rho_x(t)$. (b) The dynamics of $\rho_y(t)$. Since all sequences converge, the attraction basin is at least equal to $0.05n$.

TABLE I
THE MEMORY CAPACITY OF PARTIAL CONNECTION SECOND-ORDER BAM

r	α'
10	0.0630
5	0.0535
2	0.0371
1	0.0217
0.5	0.00928
0.2	0.00214
0.1	0.000630

case of partial-order-connection g-order BAM, where g is a positive integer constant, the connections from F_X to F_Y are

$$u_{k,i_1,i_2,\cdots,i_g} = \sum_{h=1}^{m} y_{kh} x_{i_1 h} x_{i_2 h} \cdots x_{i_g h} \tag{40}$$

where $k = 1, \cdots, p$, $i_1 = 1, \cdots, n-g+1$, $i_2 = i_1+1, \cdots, n-g, \cdots$, and $i_g = i_{g-1} + 1, \cdots, n$. The connections from F_Y to F_X are

$$v_{j,l_1,l_2,\cdots,l_g} = \sum_{h=1}^{m} x_{jh} y_{l_1 h} y_{l_2 h} \cdots y_{l_g h} \tag{41}$$

where $j = 1, \cdots, n$, $l_1 = 1, \cdots, p-g+1$, $l_2 = l_1 + 1, \cdots, p-g$, $\cdots$, and $l_g = l_{g-1} + 1, \cdots, p$. The corresponding recalling rules are

$$y_k^{(t+1)} = \mathrm{sgn}\left(\sum_{i_g, i_{g-1}, \cdots, i_1} u_{k,i_1,i_2,\cdots,i_g} x_{i_1}^{(t)} x_{i_2}^{(t)} \cdots x_{i_g}^{(t)} \right) \tag{42}$$

TABLE II
THE ATTRACTION BASIN AND THE FINAL ERROR IN THE RETRIEVAL FOR THE PARTIAL CONNECTION SECOND-ORDER BAM WHEN $r = 1$

α	$\rho_{maxinit,r=1}$	$\rho_x^{final} = \rho_y^{final}$
0.02	5.79×10^{-2}	7.57×10^{-3}
0.018	7.86×10^{-2}	3.11×10^{-3}
0.016	9.76×10^{-2}	1.19×10^{-3}
0.014	1.16×10^{-1}	3.70×10^{-4}
0.012	1.36×10^{-1}	8.19×10^{-5}
0.010	1.56×10^{-1}	1.02×10^{-5}
0.008	1.80×10^{-1}	4.51×10^{-7}
0.006	2.06×10^{-1}	2.44×10^{-9}

$$x_j^{(t+1)} = \mathrm{sgn}\left(\sum_{l_g, l_{g-1}, \cdots, l_1} v_{j,l_1,l_2,\cdots,l_g} y_{l_1}^{(t)} y_{l_2}^{(t)} \cdots y_{l_g}^{(t)} \right). \tag{43}$$

where

$$i_1 = 1, \cdots, n-g+1,\ i_2 = i_1 + 1, \cdots, n-g, \cdots\cdots,$$
$$\text{and } i_g = i_{g-1} + 1, \cdots, n$$

and

$$l_1 = 1, \cdots, p-g+1,\ l_2 = i_1 + 1, \cdots, p-g, \cdots\cdots,$$
$$\text{and } l_g = i_{g-1} + 1, \cdots, p.$$

Note that

$$E\left[\left(x_{jh} \sum_{l_g, l_{g-1}, \cdots, l_1} y_{l_1 h} y_{l_2 h} \cdots y_{l_g h} \right)^2 \right] = \binom{p}{g} \tag{44}$$

and

$$E\left[\left(y_{kh} \sum_{i_g, i_{g-1}, \cdots, i_1} x_{i_1 h} x_{i_2 h} \cdots x_{i_g h} \right)^2 \right] = \binom{n}{g}. \tag{45}$$

Following the method presented in Section IV, one can estimate the lower bound of memory capacity of the partial-order-connection case. That is

$$\min\left(\frac{n^g}{3gg! \log n}, \frac{p^g}{3gg! \log p} \right). \tag{46}$$

Also, if we follow the methodology of Section V and assume that $m = rn^g$, we can obtain the following theorems for the partial-order-connection g-order BAM.

Theorem 5: For large n and p

$$P_Y^{**} \leq 1 - \exp\left\{ n\hbar\left(\rho_x^{(t)}\right) + \log \alpha n^g + rn\hbar(\rho_y) - \frac{\rho_y r \mathrm{poly}_1\left(n^{2g+1}, {\rho_x^{(t)}}^{2g}\right)}{c_2 \mathrm{poly}_2(n^{2g})} \right\}$$

$\mathrm{poly}_1(\cdot,\cdot)$ is a polynomial of n and $\rho_x^{(t)}$ in which the highest order term is $c_1 n^{2g+1} {\rho_x^{(t)}}^{2g}$, where c_1 is a function of g. Also, $\mathrm{poly}_2(\cdot)$ is a polynomial of n with degree $2g$ and c_1 is a function of g.

TABLE III
THE ATTRACTION BASIN AND THE FINAL ERROR IN THE RETRIEVAL FOR THE PARTIAL CONNECTION SECOND-ORDER BAM WHEN $r = 2$

α	$\rho_{maxinit,r=2}$	ρ_x^{final}	ρ_y^{final}
0.02	1.27×10^{-1}	7.14×10^{-5}	1.43×10^{-4}
0.018	1.39×10^{-1}	2.34×10^{-5}	4.68×10^{-5}
0.016	1.51×10^{-1}	5.82×10^{-6}	1.16×10^{-5}
0.014	1.65×10^{-1}	9.75×10^{-7}	1.95×10^{-6}
0.012	1.80×10^{-1}	9.02×10^{-8}	1.84×10^{-7}
0.010	1.96×10^{-1}	3.21×10^{-9}	6.04×10^{-9}
0.008	2.15×10^{-1}	2.16×10^{-11}	4.33×10^{-11}
0.006	2.37×10^{-1}	5.27×10^{-15}	1.05×10^{-14}

Theorem 6: For large n and p

$$P_X^{**} \le 1 - \exp\left\{ p\hbar\left(\rho_y^{(t+1)}\right) + \log\alpha\left(\frac{p}{r}\right)^g + \frac{p}{r}\hbar(\rho_x) - \frac{\rho_x r^{g-1}\text{poly}_3\left(p^{2g+1}, {\rho_y^{(t)}}^{2g}\right)}{c_4\text{poly}_4(p^{2g})} \right\}$$

$\text{poly}_3(\cdot,\cdot)$ is a polynomial of p and $\rho_y^{(t)}$ in which highest order term is $c_3 p^{2g+1} {\rho_y^{(t)}}^{2g}$, where c_3 is a function of g. Also, $\text{poly}_4(\cdot)$ is a polynomial of p with degree $2g$ and c_4 is a function of g.

Note that the above polynomials depends on g. For a given g, one can use the above two theorems to construct the dynamics of partial-order-connection g-order BAM.

In case of the total-order-connection g-order BAM, the connections from F_X to F_Y are

$$u_{k,i_1,i_2,\cdots,i_g} = \sum_{h=1}^{m} y_{kh} x_{i_1 h} x_{i_2 h} \cdots x_{i_g h} \tag{47}$$

where $k = 1,\cdots,p$, $i_1 = 1,\cdots,n$, $i_2 = 1,\cdots,n$, $\cdots,\cdots$, and $i_g = 1,\cdots,n$. The connections from F_Y to F_X are

$$v_{j,l_1,l_2,\cdots,l_g} = \sum_{h=1}^{m} x_{jh} y_{l_1 h} y_{l_2 h} \cdots y_{l_g h} \tag{48}$$

where $j = 1,\cdots,n, l_1 = 1,\cdots,p, l_2 = 1,\cdots,p,\cdots, \cdots$, and $l_g = 1,\cdots,p$. The corresponding recalling rules are

$$y_k^{(t+1)} = \text{sgn}\left(\sum_{i_g,i_{g-1},\cdots,i_1}^{n} u_{k,i_1,i_2,\cdots,i_g} x_{i_1}^{(t)} x_{i_2}^{(t)} \cdots x_{i_g}^{(t)} \right) \tag{49}$$

$$x_j^{(t+1)} = \text{sgn}\left(\sum_{l_g,l_{g-1},\cdots,l_1}^{p} v_{j,l_1,l_2,\cdots,l_g} y_{l_1}^{(t1)} y_{l_2}^{(t1)} \cdots y_{l_g}^{(t1)} \right). \tag{50}$$

We can obtain similar results for the total connection g-order BAM based on the following lemma.

Lemma 6: Let x_i be ± 1 equiprobable independent random variables

$$E\left[\left(\sum_{i=1}^{n} x_i\right)^{2g}\right] \le \frac{(2g)!}{2^g g!} n^g$$

where g is an integer.

Proof: Let Z be a normal random variable with zero mean and variance being σ^2. The kth moment of Z [17] is

$$E[Z^k] = \begin{cases} 0, & k \text{ is odd} \\ 1\cdot 3\cdots(k-1)\sigma^k, & k \text{ is even} \end{cases} \tag{51}$$

where k is a positive integer. Also, let z_i's be independent standard normal random variables. Then,

$$E[z_i^k] = \begin{cases} 0, & k \text{ is odd} \\ 1\cdot 3\cdots(k-1), & k \text{ is even.} \end{cases}$$

Note that $\sum_{i=1}^{n} z_i$ is a normal random variable with zero mean and variance n.

Clearly

$$E[x_i^k] \le E[z_i^k]$$

for every k.

Thus

$$\begin{aligned} E\left[\left(\sum_{i=1}^{n} x_i\right)^{2g}\right] &\le E\left[\left(\sum_{i=1}^{n} z_i\right)^{2g}\right] \\ &= 1\cdot 3\cdots(2g-1)n^g \\ &= \frac{(2g)!}{2^g g!} n^g \end{aligned} \tag{52}$$

Q.E.D.

VII. CONCLUSION

We have studied several properties of second-order BAM. The properties are stability, capacity and statistical dynamics.

- We have given an example to show that the state of second-order BAM may converge to limited cycles.
- When error in the retrieval pairs is not allowed, the lower bound of memory capacity is

$$O\left(\min\left(\frac{n^2}{\log n}, \frac{p^2}{\log p}\right)\right).$$

- When error in the retrieval pairs is allowed, we have introduced a methodology to estimate the dynamics of second-order BAM. Based on this dynamics, we can estimate the memory capacity, attraction basin and final error in the retrieval pairs.

Following the methodology presented in this paper, we can analyze the properties of general higher-order BAM. The main advantage of the methodology is that we can analyze some associative memories which admit limited cycles.

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Psychophysical 1-D Wavelet Analysis and the Appearance of Visual Contrast Illusions

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***Abstract*—Psychophysical representations built by Gabor visual channels described in complex analytic form are shown to be related to the wavelet transform of visual stimuli under empirically plausible bandwidth constraints. Analysis of the psychophysical wavelet representations of one-dimensional stimuli eliciting some visual contrast illusions (Mach bands and the Craik–Cornsweet–O'Brien illusion) reveals that qualitative aspects of the selective appearance of these illusions can be explained as a natural consequence of the functional characteristics of early visual processing.**

I. Introduction

Since the work of Campbell and Robson [8], overwhelming psychophysical evidence has accumulated over the past quarter-century which suggests that early spatial visual processing is performed by a bank of two-dimensional (2-D) filters (or visual channels) each of which is selectively sensitive to a narrow range of spatial frequencies and orientations. These channels are conceived of as working locally and in parallel for the analysis of spatial patterns, in much the same way as cortical simple cells in cats [35] and monkeys [53] do. Since visual channels are the building block for a theoretical description of early visual processing, a complete characterization of this processing stage requires i) determining their structural and functional characteristics and ii) defining the nature of the early spatial representations that they produce. Besides the complete characterization of early visual processing, a satisfactory understanding of spatial vision requires the provision of late visual processing models stating how visual representations are further processed in order to achieve visual tasks [20].

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Although early visual processing has been modeled using a number of different approaches (e.g., coding by local features [3], coding by edges or Laplacian zero-crossings [37], multiscale analysis [55], coding by pure spatial-frequency components [43]), the notion is widely accepted now that early visual processing produces a transformation of the input image which yields a joint spatial/spatial-frequency representation that is formally similar to those used in various fields of theoretical physics [32] and signal analysis [29]. The notion of a psychophysical joint representation has been commonplace in auditory research [34], [38], but it has taken much longer to settle with the vision research community.

This paper analyzes the consequences of early visual processing on the appearance of some visual contrast illusions. Visual illusions are interesting to analyze because they should be explained as a result of the operation of the same mechanisms and processes which account for empirical data on nonillusory (or "veridical") perception. Therefore, explaining the appearance of visual illusions is not a separate research problem, and the success (or lack thereof) at accounting for them may support (or reject) a specific formal characterization of visual processing. In Section II, visual channels are described as Gabor filters in complex analytic form [15], [16], [20], [44]. Section III shows that, given some empirically plausible bandwidth constraints, the bank of visual channels comprise a family of affine coherent states and, therefore, the visual representation built by those channels is a transformation of the joint spatial/spatial-frequency representation known as wavelet transform. Section IV derives and analyzes the wavelet transform of some elementary nonstationary features (singularities and discontinuities). Finally, Section V shows that the wavelet transform of luminance profiles eliciting contrast illusions (Mach bands and the Craik–Cornsweet–O'Brien illusion) contain the signatures of the illusory features, thus explaining the appearance of these illusions as a necessary consequence of early spatial visual processing.

II. Functional Characteristics of Visual Channels

Although the formal characterization of 2-D visual channels must be done in 2-D, a simpler one-dimensional (1-D) characterization will be sufficient for the purpose of this paper, where only images consisting of 1-D luminance profiles (1-D images) will be considered. Under these circumstances, a 2-D visual channel is completely characterized by its sensor line-weighting function (LWF). Empirical constraints and theoretical considerations led several authors [1], [13], [25], [54] to propose that the LWF at spatial position x' of the channel tuned to spatial frequency u' has the mathematical form of the elementary signal which Gabor [18] showed to minimize the space/spatial-frequency uncertainty relation. Expressed in complex analytic form, this LWF is

$$\psi^*(x; x', u') = \frac{\gamma(u')}{\sqrt{2\pi} s_x} \exp\left[-\frac{(x-x')^2}{2s_x^2}\right] e^{-i2\pi u'(x-x')}$$
$$x', u' \in \mathbb{R}, \quad u' \neq 0 \qquad (1)$$

where the asterisk denotes complex conjugation, u' is the channel's tuning frequency, $\gamma(u')$ is its gain, and $s_x > 0$ is the standard deviation of the modulating Gaussian determining the spatial spread of the LWF (see Fig. 1(a) and (b)). The 1-D transfer function of the u' c/deg channel at location x' is the Fourier transform of ψ, $\hat{\psi}$, and is given by (see Fig. 1(c))

$$\hat{\psi}(u; x', u') = \gamma(u') \exp[-2\pi^2 s_x^2 (u-u')^2] e^{-i2\pi u x'}. \qquad (2)$$